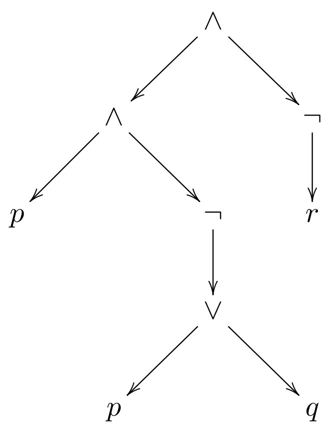
Exercises for Chapter I.1

Exercise 1 (Strict Formulas, Prioritized Formulas)
Which of the following expressions are strict formulas?

1. x
2. $(x$
3. (x)
4. $(x \Rightarrow (y \wedge z)$
5. $(x \Rightarrow (y \wedge z))$
6. $\neg(x \vee y)$
7. $(\neg(x \vee y))$
8. $\neg(x \Rightarrow y) \vee \neg(y \wedge z)$
9. $\neg(x)$
10. $(\neg((x \vee y) \wedge z) \Leftrightarrow (u \Rightarrow v)))$

Exercise 2 (✑ Formulas and Priority)
Prove that every (strict) formula is a prioritized formula.

Exercise 3 (Formulas and Priority)
Consider the following tree representation of the formula $((p \wedge \neg(p \vee q)) \wedge \neg r)$.



Using priority, this formula can be written with fewer parentheses: $p \wedge \neg(p \vee q) \wedge \neg r$ using logical notation, $p\ \overline{p+q}\ \overline{r}$ using boolean notation. Similarly, give the tree representation, logical notation and boolean notation, both using as few parentheses as possible of the following formulas:

1. $(\neg(a \wedge b) \Leftrightarrow (\neg a \vee \neg b))$.
2. $((\neg a \vee b) \wedge (\neg b \vee a))$.
3. $(((a \wedge b) \wedge c) \vee ((\neg a \wedge \neg b) \wedge \neg c))$.
4. $(p \Rightarrow (q \Rightarrow r))$.
5. $((p \Rightarrow q) \Rightarrow r)$.

Exercise 4 (Formulas and Priority)
Give the tree representation of the following prioritized formulas:

1. $p \Leftrightarrow \neg q \lor r$.
2. $p \lor q \Rightarrow r \land s$.
3. $p \lor q \Rightarrow r \Leftrightarrow s$.
4. $p \lor q \land r \Rightarrow \neg s$.
5. $p \Rightarrow r \land s \Rightarrow t$.
6. $p \lor q \land s \lor t$.
7. $p \land q \Leftrightarrow \neg r \lor s$.
8. $\neg p \land q \lor r \Rightarrow s \Leftrightarrow t$.

Exercise 5 (Height of a Formula,*)
We define the height h of a prioritized formula as:

- $h(\top) = 0$ and $h(\bot) = 0$.
- If A is a variable $h(A) = 0$.
- $h(\neg A) = 1 + h(A)$.
- $h((A)) = h(A)$.
- $h(A \circ B) = max(h(A), h(B)) + 1$, if $\circ$ is one of the following operations: $\lor, \land, \Rightarrow, \Leftrightarrow$.

1. Show that this definition is ambiguous, that is, show that there exists at least one formula (by presenting one) that can have two different heights under this definition.
2. Give a new recuseive definition of height of a formula that does not have this problem.

Exercise 6 (Validity)
Give the truth table of the following formulas:

1. $p \Rightarrow (q \Rightarrow p)$.
2. $p \Rightarrow (q \Rightarrow r)$.
3. $(p \Rightarrow q) \Rightarrow (p \Rightarrow r)$.
4. $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$.

Indicate which formulas are valid, and which are equivalent.

Exercise 7 (✒ Circular Reasoning)
Give the truth table of the following formulas: $a \Rightarrow b$, $b \Rightarrow c$, $c \Rightarrow a$, $a \Leftrightarrow b$, $b \Leftrightarrow c$, and $c \Leftrightarrow a$. Conclude that $(a \Rightarrow b) \land (b \Rightarrow c) \land (c \Rightarrow a) \equiv (a \Leftrightarrow b) \land (b \Leftrightarrow c) \land (c \Leftrightarrow a)$.

Exercise 8 (Equivalence)
Give the truth table of the following formulas:

1. $p \land (p \lor q)$.
2. $\neg p \land \neg q$.
3. $\neg(p \land q)$.
4. $\neg(p \lor q)$.
5. $p \lor (p \land q)$.
6. $\neg p \lor \neg q$.
7. p.

Indicate which formulas are equivalent.

Exercise 9 (Equivalence)
Give the truth table of the following formulas:

1. $(p \Rightarrow q) \wedge (q \Rightarrow p)$.
2. $p \Rightarrow q$.
3. $p \Leftrightarrow q$.
4. $\neg q \Rightarrow \neg p$.
5. $(p \wedge q) \vee (\neg p \wedge \neg q)$.

Indicate which formulas are equivalent.

Exercise 10 (Equivalence)
Which of the following formulas are equivalent to $p \Rightarrow q \vee r$?

1. $q \wedge \neg r \Rightarrow p$.
2. $p \wedge \neg r \Rightarrow q$.
3. $\neg q \wedge \neg r \Rightarrow \neg p$.
4. $q \vee \neg p \vee r$.

Exercise 11 (✒ Model for a set of formulas,*)
Show that an assignment is a model for a set of formulas if and only if it is a model for the conjunction of all the formulas in the set.

Exercise 12 (✒ Simplification Laws)
Prove that for any x, y

- $x \vee (x \wedge y) \equiv x$.
- $x \wedge (x \vee y) \equiv x$.
- $x \vee (\neg x \wedge y) \equiv x \vee y$.

Exercise 13 (✒ Simplification of Formula,*)
Show by simplification that the following formula is a tautology.

$$(a + b) \cdot (\bar{b} + c) \Rightarrow (a + c)$$

Exercise 14 (Models and Normal Forms)
Let A be the following formula:

$$(((a \Rightarrow \neg b) \Leftrightarrow \neg c) \wedge (c \vee d)) \wedge (a \Leftrightarrow d).$$

1. Is A a tautology? (justify)
2. Is A a contradiction ? (justify)
3. Give the conjunctive normal form for A.
4. Give the disjunctive normal form for A.

Exercise 15 (✒ Natural induction,*)
Show that if a formula contains only one variable and the operations $\vee$ and $\wedge$ (no negation), then it is equivalent to a formula of size 0.

Exercise 16 (Boolean Algebra)
Using truth tables, determine if the operations $\Rightarrow, \Leftrightarrow$ are commutative, associative, idempotent, transitive.

Exercise 17 (Boolean Algebra)
Show that a formula with only one variable (say, that variable is p) is equivalent to either 0, 1, p, or $\neg p$.

Exercise 18 (Boolean Algebra)
Write 16 formulas such that any formula with two variables, say p and q, is equivalent to one of those 16 formulas.

Exercise 19 (Boolean Function)
Determine the number of distinct (non-equivalent) boolean functions with k arguments (express your answer as a function of k).

Exercise 20 (Consequence)
During an inquiry, adjutant Tinet makes the following reasoning:

- If the murder occurred during the day, then the murderer is a friend of the victim
- However, the murder occurred at night

Therefore, the murderer is not a friend of the victim. Is adjutant Tinet correct? To determine this, proceed in three steps:

1. Formalize the facts
2. Formalize the reasoning that allows to arrive to the conclusion from the hypotheses
3. Determine if the reasoning is correct.

Exercise 21 (Consequence)
Pinnochio, Quasimodo and Romeo are singing in a choir. They decide that:

1. If Pinocchio does not sing, then Quasimodo will.
2. If Quasimodo sings, then so will Pinocchio and Romeo.
3. If Romeo sings, then at least one of Quasimodo or Pinocchio will not.

Determine whether Pinocchio will sing. Justify your answer by formalizing the reasoning.

Exercise 22 (Consequence)
Formalize the following statements using logic connectors.

(a) If Peter went home, then John went to the cinema.
(b) Mary went to the library or Peter went home.
(c) If John went to the cinema, then Mary went to the library or Peter went home.
(d) Mary is not at the library and John went to the cinema.
(e) Peter went home.

Is the last statement a consequence of the preceding ones?

Exercise 23 (Normal Forms)
For each of the following formulas, write the equivalent disjunctive normal form, and prove whether or not it is satisfiable (by giving a model for the formula if possible).

- $\neg(a \Leftrightarrow b) \vee (b \wedge c) \Rightarrow c$.
- $(a \Rightarrow b) \wedge (b \Rightarrow \neg a) \wedge (\neg a \Rightarrow b) \wedge (b \Rightarrow a)$.

Exercise 24 (Normal Forms)
Let A be the formula $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$. Find the formula in disjunctive normal form equivalent to $\neg A$. Is A valid?

Exercise 25 (✐ Normal Forms,)**
We come back to the proofs regarding the transformation of formulas into normal forms.

1. Show that the repeated application of the elimination of equivalence, elimination of implications and displacement of negations on a formula yields a formula in normal form.
2. Show also regardless of the order in which the elinimation of equivalence, elimination of implications and displacement of negations are used, the algorithm for transforming a formula in normal form will terminate.

Exercise 26 (Consequence and normal forms)
Adjutant Tinet is on a new inquiry. His hypotheses are as follows:

- If John did not meet Peter the other night, then Peter is the murderer or John is a liar.
- If Peter is not the murderer, then John did not meet Peter the other night and the crime occurred after midnight.
- If the crime occurred after midnight, then Peter is the murderer or John is a liar.

He concludes that Peter is the murderer. Is his reasoning correct? Give your answer by constructing the conjunction of the hypotheses and the *negation of the conclusion*, and by putting this disjunction in conjunctive normal form. Recall that a reasoning is incorrect if and only if one of the monomials does not have any complementary litterals: this monomial gives a model of the hypotheses that is a counter-model of the conclusion.

Exercise 27 (Formalization, sum of monomials*)
On an isolated island, the natives are split into two tribes: the Tame, and the Lame. The Tame always tell the truth, the Lame always lie. We meet two natives: Aha and Beeby[1]

(a) Aha says: “ At least one of us is a Lame ”. Can we deduce Aha and Beeby's tribe?
(b) Aha says: “ At most one of us is a Lame ”. Can we deduce Aha and Beeby's tribe?
(c) Aha says: “ Both of us are in the same tribe ”. Can we deduce Aha and Beeby's tribe?

Exercise 28 (Complete and incomplete set of connectors,)**
A set of constants and connectors is called *complete* if any boolean function can be expressed with these connectors. Theorem 6.4 page 25 shows thaat the set $\{0, 1, +, -, \cdot\}$ is complete.

1. Show that the set $\{+, -\}$ is complete.
   Hint: it suffices to show that $\{0, 1, \cdot\}$ can be defined only with $+$ and $-$.
2. Show that the set $\{0, \Rightarrow\}$ is complete.
3. Let $\mid$ be the following operation: $x \mid y$ is true if and only if neither x nor y is true, that is $x \mid y$ is true only if $x = 0$ and $y = 0$. This operation is also called *NOR* (for “not OR”) because it is the negation of the “OR” ($\vee$) connector.
   Show that $\{\mid\}$ is complete.
4. (**) Show that the set $\{0, 1, +\cdot\}$ is not complete.
   Hint: boolean function that are defined using only these operations have a property that is not true of all boolean functions.
   We say that a boolean function is *monotone* (or *monotonic*) if whenever $a_1 \leq b_1, \ldots, a_n \leq b_n$, then $f(a_1, \ldots, a_n) \leq f(b_1, \leq, b_n)$.
   Show that every boolean function defined only with $\{0, 1, +, \cdot\}$ is monotone.
   Give a boolean function that is not monotone. Conclude that $\{0, 1, +, \cdot\}$ is not complete.

[1]This problem comes from a book by Raymond M. Smullyan : “ What is the Name of this Book ”, which describes many other amusing problems with these natives.

5. (*) Is the set $\{1, \Rightarrow\}$ complete?
6. (**) Is the set $\{0, \Leftrightarrow\}$ complete?

Exercise 29 (↬ Duality, recurrence,)**
Consider formulas that only contains the operations $0, 1, \neg, \wedge, \vee$. The dual of one of these formulas is obtained by turning conjunctions into disjunctions, 0 into 1 and vice versa. We write A^* to denote the dual of formula A.

1. Define A^* y recurrence on these formulas.
2. Show that if 2 formulas are equivalent, then so are their duals.
   Hint: let v be an assignment, and let $\overline{v}$ be the assignment complementary to v (0 becomes 1, 1 becomes 0).
   (a) Show that for every formula A above, $[A^*]_v = \overline{[A]_{\overline{v}}}$.
   (b) Deduce that if 2 such formulas are equivalent, then so are their duals.
3. Deduce from the preceding question that if a formula is valid, its dual is contradictory.

Exercise 30 (Canonical Form,*)**
Let $x_{i(0 \leq i \leq n)}$ be a list of n distinct variables. In this exercise, all the variables used in formulas are in this list. To any formula A, we associate a boolean function $f(x_1, \ldots, x_n) = A$. This means that for $a_1, \ldots a_n \in \mathbb{B}$, the value of $f(a_1, \ldots, a_n)$ is the value taken by A when $x_1 = a_1, \ldots, x_n = a_n$. We say that two formulas A and B are in the same class if A and B are equivalent.

1. How many classes of formulas (with our n variables) are there?
2. We denote by $\oplus$ le exclusive-or operation. We have $x \oplus y = 1$ if and only if $x = 1$ or $y = 1$, but not both (hence the name exclusive-or). We can also define this operation as $x \oplus y = \neg(x \Leftrightarrow y)$. It should be clear that this operation is commutative, associative, that $\wedge$ distributes on $\oplus$ and that 0 is the neutral element for $\oplus$. Express negation, disjunction, implication and equivalence as an exclusive-or sum of 1's and conjunctions or variables (i.e. like a formula in disjunctive normal form, except that the $\oplus$ is used instead of $\vee$).
3. Show that every boolean formula is equivalent to an exclusive-or sum of 1's and conjunctions or variables.
4. As a convention, the product of zero variables is considered to be 1 and the exclusive-or sum of zero variables is 0. Show that every boolean formula is equivalent to an exclusive-or sum of products such that not two monomials have the same set of variables.
5. Fix an arbitrary ordering of the n variables used in the formulas (when variables are identifiers, we often use the lexicographic order). An exclusive-or sum of products of variables is in Reed-Muller Canonical Form if and only if:
   - the products do not contain any repeated variables, and inside each products, the variables are ordered from left to right in increasing order,
   - the products are orderes from left to right in inverse lexicographic order deduced from the ordering of the variables.

   Show that every boolean formula is equivalent to a formula in Reed-Muller Cacnonical Form. Deduce that the expression of a formula in Reed-Muller Canonical form is unique.

Exercises for Chapter I.2

Exercise 31 (Resolvent)
Through resolution, we have:

$$\frac{a+b \quad \overline{a}+\overline{b}}{b+\overline{b}}$$

Show that: $b+\overline{b} \not\models (a+b).(\overline{a}+\overline{b})$.

Exercise 32 (Resolvent)
We recall that two clauses are equal if and only if they have the sets of literals.

- Are the clauses $p+q+\overline{r}+q+p+s+q+\overline{r}$ and $s+q+\overline{r}+p$ equal?
- Are the clauses $p+q+r+p$ and $q+r+q+r+q+r$ equal? Is one included in the other? Is one the consequence of the other? Are they equivalent?
- Give all the resolvents of the clauses $a+\overline{b}+c$ and $a+b+\overline{c}$. Are these resolvents valid?

Exercise 33 (Proof)
The following sets of formulas are unsatisfiable.

- $\{a, a \Rightarrow b, \overline{b}\}$.
- $\{a+b, \overline{a}+c, \overline{a}+\overline{d}, d+\overline{c}, \overline{b}+a\}$.
- $\{a+b+c, \overline{a}+b, \overline{b}+c, \overline{c}+a, \overline{a}+\overline{b}+\overline{c}\}$.

Prove that fact using resolution.

Exercise 34 (Formalization and Resolution,*)
We note that: " x unless y " can be formalized as $\overline{x \Leftrightarrow y}$. In a haunted house, ghosts show their presence in two ways: **obscene chants** and **sardonic laughs**. However, we can influence their behaviour by playing the organ or burning incense. Given the following:

(i) The ghosts don't do obscene chants unless we play the organ while the ghosts are not laughing.
(ii) If we burn incense, the ghosts laugh if and only if they are also chanting.
(iii) At the moment, the ghosts are chanting, but not laughing.

And our conclusion:

(iv) At the moment, we are playing the organ, but not burning incense.

We define the following:

- c: the ghosts are chanting.
- o: we are playing the organ.
- l: the ghosts are laughing.
- i: we are burning incense.

1. Transform the expression $\overline{x \Leftrightarrow y}$ into a product of clauses.
2. Formalize the hypotheses and the negation of the conclusion into a product of clauses.
3. Using resolution, prove that the reasoning is correct.

In other words, transform the conjunction of the hypotheses and the negation of the conclusion into a product of clauses and deduce the empty clause.

Exercise 35 (Proof,*)
Show, using a proof by resolution, the correctness of the following reasoning:

$$r + q \Rightarrow t, t.q \Rightarrow r, q \models t \Leftrightarrow r.$$

Exercise 36 (Formalization and Proof,*)
Show, using resolution, that the following reasoning is correct:

- The weather is nice unless it is snowing.
- It is raining unless it is snowing.
- The weather is nice or it is raining.
- Hence, it is not snowing.

Exercise 37 (✒ Defining a Clause)
A clause is either the empty clause, or a (non-empty) disjunction of literals. Give a inductive definition of a clause and define by recurrence the function s such that $s(C)$ is the set of literals in clause C.

Exercise 38 (✒ Proof)
Prove the property 2.1.14 page 44 on the monotony and composition.

Exercise 39 (✒ Property of Resolution)
Prove property 2.1.8 page 43.

Exercise 40 (✒ Resolution does not add literals,*)
Let Γ be a set of clauses. A literal in Γ is a literal that appears in one of the clauses of Γ. Show that any clause deduced from Γ only contains literals in Γ.

Exercise 41 (Formalization and resolution)
Consider the following hypotheses:

1. If Peter misses his tournament then Peter will be depressed.
2. If the weather is nice then Peter will go to the swimming pool.
3. If Peter does not go to the swimming pool he will be depressed.
4. When at the swimming pool, Peter does not train.
5. Peter will miss his tournament if he does not train.

We would like to demonstrate that under such hypotheses, the following conclusion can be derived:

6. Peter will be depressed.

You will proceed as follows:

- Formalize the hypotheses and the negation of the conclusion.
- Deduce a set of clauses equivalent to your formal statements.

- Show that the conclusion can be correctly derived from the hypotheses, by proving using the resolution method that the set of clauses is contradictory.

Exercise 42 (Formalisation and resolution)

The Beatles were a rock band formed in the 60's in Liverpool. It was composed of four boys: Ringo Starr, Paul MacCartney, John Lennon and Georges Harrison. It has been quite hard to decide who would play which instrument, as the following extract from their discussion will show:

Paul said: "If Ringo doesn't play the guitar, then I will play the bass guitar and John the guitar."

Georges said: "I will play the guitar if, and only if, John plays it too."

John said: "If Paul plays the bass guitar, then Georges will play the guitar."

Ringo said: "I will play the drums and therefore, not the guitar."

After their talk, they settled on the following power four:

- Ringo would play the drums,
- Paul would play the bass guitar
- John and Georges would both play the guitar.

We will now show that every member of the band is happy with this conclusion.

1. Formalise the 4 hypotheses and the conclusion using the following propositional variables:
   - RD: "Ringo plays the drums",
   - RG: "Ringo plays the guitar",
   - PB: "Paul plays the bass guitar",
   - JG: "John plays the guitar",
   - GG: "Georges plays the guitar".
2. Transform the hypotheses and the negation of the conclusion into clauses.
3. Prove, using resolution, that the conclusion is a consequence of the hypotheses.

Exercise 43 (Reduction, X1603 Andrews)
Soit l'ensemble de clauses :

$$\{p + q, \overline{p} + r + \overline{q} + p, p + \overline{r}, q + \overline{p} + \overline{q}, q + \overline{r} + p, r + q + \overline{p} + \overline{r}, \overline{r} + q\}.$$

1. Reduce this set (reduction is defined in 2.1.4 page 48).
2. Determine whether this set is satisfiable using resolution.

Exercise 44 (DPLL)
Consider the following set of clauses:

$$\{\overline{a} + \overline{b} + \overline{f}, a + b + f, e + \overline{a}, \overline{a} + \overline{b}, \overline{a} + c, d + a + \overline{d}, a + b, \overline{a} + \overline{c} + \overline{d}, d\}.$$

- Use algorithm Algo_DPLL on this set of clauses.
- Conclude whether or not it is satisfiable.
- If it is satisfiable, give a model obtained from the trace of the algorithm.

In the tree of calls, identify the steps as follows:

- Elimination of valid clauses (VAL).
- Reduction (RED).
- Pure literal elimination (PLE).
- Unit resolution (UR).

In addition, note the assignments made at each step of the algorithm so that the model can easily be recovered.

Exercise 45 (DPLL)
Use algorithm Algo_DPLL to determine whether or not the following sets of clauses are satisfiable:

- $\{a+b+c+d+e+f, \overline{a}+b, \overline{b}+a, \overline{c}+d, \overline{d}+c, \overline{b}+\overline{c}, \overline{b}+c, b+\overline{c}, \overline{e}, \overline{f}\}$.
- $\{a+b+c+d+f, \overline{a}+b, \overline{b}+a, \overline{c}+d, \overline{d}+c\}$.
- $\{b+j+\overline{a}, a+j+\overline{b}, b+a+j, a+j, j+b, \overline{b}+\overline{j}, \overline{j}+b, j+s, \overline{s}+\overline{b}\}$.
- $\{a+\overline{c}+d, \overline{b}+c+f, b+\overline{e}+f, \overline{c}+e+\overline{f}, e+f, c+d, \overline{a}, \overline{e}+\overline{f}\}$.

Give a trace of the algorithm.

Exercise 46 (Complete Strategy)
Let Γ be the following product of clauses:

$$(a+b+\overline{c}).(b+\overline{c}).(c+\overline{c}).(b+c).(\overline{a}+\overline{b}+c).(a+\overline{b}+c).$$

Determine using the complete resolution strategy whether Γ is unsatisfiable, or if it has a model. Give a trace of the algorithm. Give the proof(s) you obtained. If Γ has a model, show it.

Exercise 47 (Complete Strategy)
Consider the following set of clauses:

$$\{p+q, \overline{p}+s, \overline{s}+t, \overline{t}, \overline{q}+r, \overline{r}, \overline{r}+p+t, q+z+\overline{z}, \overline{q}+r+s\}.$$

Use the complete strategy on this set of clauses and determine whether it is satisfiable or not.

Exercise 48 (Complete Strategy)
Consider the function f such that $f(x,y,z)=0$ if and only if there is an even number of arguments with value 1. Express f as a product of clauses using the method described in subsection 1.6 page 32, then simplify f using the complete strategy.

Exercise 49 (✎ Proof)
Prove Lemma 2.3.2 page 55.

Exercise 50 (✎ Proof)
Prove Lemma 2.3.5 page 55.

Exercise 51 (From SAT to 3-SAT,*)**
SAT (short for Satisfiability) is a decision problem stated as follows: "given a set of clauses Γ, determine if Γ has a model". 3-SAT is a restriction of this problem in which each clause in Γ contains exactly three literals.

SAT is an NP-complete problem [5]. In this exercise, we study a polynomial-time reduction from SAT to 3-SAT, thereby showing that 3-SAT is also NP-complete[2].

A polynomial-time reduction from SAT to 3-SAT is a polynomial-time transformation of the set of clauses Γ into a set of clauses Γ' verifying the following two properties:

(a) all the clauses in Γ' contain three distinct literals.
(b) Γ has a model if and only if Γ' has a model.

Usually, such reductions also satisfy the following additional property:

(c) Any model for Γ' is also a model for Γ.

The goal of this exercise is to show that the polynomial-time transformation below verify these three properties. Note that the transformation potentially introduces new variables.

Let $\Gamma = \{c_1, \ldots, c_m\}$ be a set of clauses. The reduction consists in replacing every clause $c_i = z_1 + \ldots + z_k$ of Γ (where each z_i is a literal) by a set of clauses C_i' constructed as follows, depending on the value of k:

$k = 1$ **:** $C_i' = \{z_1 + y_1 + y_2,\ z_1 + y_1 + \overline{y_2},\ z_1 + \overline{y_1} + y_2,\ z_1 + \overline{y_1} + \overline{y_2}\}$ where y_1 and y_2 are new variables (not present anywhere else).

$k = 2$ **:** $C_i' = \{z_1 + z_2 + y_1,\ z_1 + z_2 + \overline{y_1}\}$ where y_1 is a new variable.

$k = 3$ **:** $C_i' = \{c_i\}$.

$k > 3$ **:** $C_i' = \{z_1 + z_2 + y_1,\ \overline{y_1} + z_3 + y_2,\ \overline{y_2} + z_4 + y_3,\ \overline{y_3} + z_5 + y_4,\ \ldots,\ \overline{y_{k-3}} + z_{k-1} + z_k\}$ where $y_1, \ldots, y_{k-3}$ are new variables.

Finally, $\Gamma' = \bigcup_{i=1}^{m} C_i'$.

By construction, the reduction certainly verify property (a). The answers to the following questions allow us to prove properties (b) and (c):

1. Show (without a truth table) that any assignment will give the same value to c_i as to the conjunction of all the clauses in C_i' when c_i consists of only one literal.
2. Show (without a truth table) that any assignment will give the same value to c_i as to the conjunction of all the clauses in C_i' when c_i consists of two literal.
3. Let c_i be a clause with more than 3 literals. Show that if c_i has a model, then C_i' also has a model.
4. Let c_i be a clause with more than 3 literals. Show that any model for C_i' is also a model for c_i.
5. The reduction above maintains satisfiability. Show that the reduction does not maintain the meaning of the set of formulas. That is, the conjunction of all the clauses in Γ is not equivalent to the conjunction of all the formulas in Γ'.

[2]We note that the k-SAT is not NP-complete for any value of k: 2-SAT can be solved in polynomial time.

Exercises for Chapter I.3

Exercise 52 (Proof Sketch)
The following sequence of lines is not a proof sketch. Find the smallest number i such that lines 1 to $i-1$ is a proof draft, but lines 1 to i is not. Give the context of lines 1 to $i-1$.

context	number	proof
	1	`Suppose` a
	2	`Suppose` b
	3	c
	4	`Thus` d
	5	`Suppose` e
	6	f
	7	`Thus` g
	8	h
	9	i
	10	`Thus` j
	11	`Thus` k
	12	l

Exercise 53 (Proof of Formulas Requiring Special Rules)
Give a proof of the following formulas:

- $a \Rightarrow \neg\neg a$,
- $\neg\neg a \Rightarrow a$,
- $a \Leftrightarrow \neg\neg a$,
- $a \vee \neg a$.

Exercise 54 (Simple Proofs of Formulas)
Give a proof of the following formulas:

- $a \Rightarrow c$ in environment $a \Rightarrow b,\ b \Rightarrow c$.
- $(a \Rightarrow b) \wedge (b \Rightarrow c) \Rightarrow (a \Rightarrow c)$.
- $(a \Rightarrow b) \Rightarrow ((b \Rightarrow c) \Rightarrow (a \Rightarrow c))$.

Exercise 55 (⚯ Abbreviations)
Let A be a formula and $ufld(A)$ be the formula obtained by replacing in A the negations and equivalences (in abbreviated form) by their definition. $ufld(A)$ is the formula obtained by "unfolding", hence the name $ufld$ chosen for the unfolding function.

- Define the function $ufld$ by recurrence.
- Show that the formulas A and $ufld(A)$ are equivalent.
- Deduce that two formulas, equal up to abbreviations, are equivalent.

Exercise 56 (Proof of Formulas)
Give a proof of the following formulas:

1. $a \Rightarrow (b \Rightarrow a)$.
2. $a \wedge b \Rightarrow a$.

3. $\neg a \Rightarrow (a \Rightarrow b)$.
4. $(a \Rightarrow (b \Rightarrow c)) \Rightarrow ((a \Rightarrow b) \Rightarrow (a \Rightarrow c))$.
5. $a \wedge b \Rightarrow b \wedge a$.
6. $a \vee b \Rightarrow b \vee a$.
7. $(a \Rightarrow (b \Rightarrow c)) \Rightarrow (a \wedge b \Rightarrow c)$.
8. $(a \wedge b \Rightarrow c) \Rightarrow (a \Rightarrow (b \Rightarrow c))$.
9. $(a \Rightarrow b) \wedge (c \Rightarrow d) \Rightarrow (a \wedge c \Rightarrow b \wedge d)$.

Exercise 57 (About implication)
Give a proof of the following formulas:

1. $a \Rightarrow b$ in the environment $\neg a \vee b$.

2. $(\neg b \Rightarrow \neg a) \Rightarrow (a \Rightarrow b)$.

Exercise 58 (Boolean algebra)
Give a proof of the following formulas:

1. $\neg(a \wedge \neg a)$.

2. $a \vee a \Rightarrow a$.

3. $a \wedge a \Rightarrow a$.

4. $a \wedge (b \vee c) \Rightarrow a \wedge b \vee a \wedge c$.

5. $a \wedge b \vee a \wedge c \Rightarrow a \wedge (b \vee c)$.

6. $a \vee b \wedge c \Rightarrow (a \vee b) \wedge (a \vee c)$.

7. $(a \vee b) \wedge (a \vee c) \Rightarrow a \vee b \wedge c$.

Exercise 59 (✑ Proof of Formulas with Environment)
Give a proof of the following formulas:

- P1 : $B \vee C$ in environment $\neg B \Rightarrow C$.
- P2 : $\neg B \vee C$ in environment $B \Rightarrow C$.
- P3 : $\neg B \vee \neg C$ in environment $\neg(B \wedge C)$.
- P4 : $\neg B$ in environment $\neg(B \vee C)$.
- P5 : $\neg C$ in the same environment $\neg(B \vee C)$.
- P6 : B in the same environment $\neg(B \Rightarrow C)$.
- P7 : $\neg C$ in the same environment $\neg(B \Rightarrow C)$.

Exercise 60 (Proof of Formulas)
Give a natural deduction proof of the following formulas:

1. $\neg(a \vee b) \Rightarrow (\neg a \wedge \neg b)$.
2. $(\neg a \wedge \neg b) \Rightarrow \neg(a \vee b)$.
3. (*) $(\neg a \vee \neg b) \Rightarrow \neg(a \wedge b)$.
4. (**) $(a \vee b) \wedge (\neg a \vee b) \Rightarrow b$.
5. (***) $\neg(a \wedge b) \Rightarrow (\neg a \vee \neg b)$.

6. (***) $(a \vee b) \wedge (\neg b \vee c) \Rightarrow a \vee c$.

Exercise 61 (Midterm 2011)
Prove the following formulae using natural deduction :

- $(p \Rightarrow \perp) \wedge (\neg p \wedge q \Rightarrow \perp) \Rightarrow \neg q$
- $((p \vee q) \vee (p \vee r)) \wedge \neg p \Rightarrow q \vee r$
- (*) $(p \Rightarrow \perp) \vee (p \wedge q \Rightarrow \perp) \Rightarrow \neg q \vee \neg p$

Exercise 62 (Midterm 2013)
Give a proof of the following formulae using the natural deduction in the table format :

- $(p \vee q) \Rightarrow (\neg p \wedge \neg q) \Rightarrow r$.
- $((p \Rightarrow q) \wedge (q \Rightarrow r)) \wedge \neg r \Rightarrow \neg p$.
- $(p \Rightarrow q) \Rightarrow ((p \wedge q) \vee \neg p)$.

Exercise 63 (Questions from various midterm exams)

1. Prove the following formula using natural deduction: $(p \vee q) \wedge (p \Rightarrow r) \Rightarrow q \vee r$
2. We consider the hypotheses:
   - (H1) : If Peter is tall, then John isn't Peter's son.
   - (H2) : If Peter isn't tall, then John is Peter's son.
   - (H3) : If John is Peter's son, then Mary is John's sister.

   Formalize and prove that from these we can deduce conclusion (C) : "Mary is John's sister or Peter is tall or both" using natural deduction.

Exercises for Chapter II.4

Exercise 64 (Structure and free variables)

For each formula hereafter, indicate its structure and its free variables.

1. $\forall x(P(x) \Rightarrow \exists y Q(x,y))$.
2. $\forall a \forall b(b \neq 0 \Rightarrow \exists q \exists r(a = b * q + r \wedge r < b))$[3].
3. $Even(x) \Leftrightarrow \exists y(x = 2 * y)$.
4. $Divides(x,y) \Leftrightarrow \exists z(y = z * x)$.
5. $Prime(x) \Leftrightarrow \forall y(Divides(y,x) \Rightarrow y = 1 \vee y = x)$.

Exercise 65 (Formalization, function and relation symbols)
Consider $\Sigma = \{b^{r2}, u^{r2}, c^{r2}, o^{r2}, r^{f0}, f^{f1}\}$, a signature with the following meaning:

- $b(x,y) := x$ is y's brother.
- $u(x,y) := x$ is y's uncle.
- $c(x,y) := x$ is y's cousin.
- $o(x,y) := x$ is older than y.
- r is Robert's nickname.
- $f(x)$ returns x's father.

Formalize the following sentences in first order logic using signature Σ.

1. Every brother of Robert's father is Robert's uncle.
2. If the fathers of two children are brothers, then these children are cousins.
3. One of Robert's cousins is younger than one of Robert's brothers.

Translate in English the following logical propositions:

1. $\nexists x\ o(x, p(x))$
2. $\forall x \forall y(f(f(x)) = f(f(y)) \Rightarrow c(x,y))$
3. $\forall x \exists y(b(x,y) \wedge o(x,y))$
4. $\exists x \exists y(b(x,y) \wedge \neg(f(x) = f(y)))$

Exercise 66 (Formalization)
Consider signature $\Sigma = \{g^{f0}, f^{f0}, P^{r2}, W^{r2}\}$, whose symbols have the following meaning:

- $g :=$ Germany's team.
- $f :=$ France's team.
- $P(x,y) := x$ played against y.
- $W(x,y) := x$ won against y.

Formalize the following sentences in first order logic using siganture Σ.

1. France's team won a match and lost a match.
2. France's and Germany's team tied.
3. A team won all its matches.

[3] In order to respect the usual notation for the Euclidean Algorithm, we exceptionally use a, b, q and r as variable names

4. No team lost all its matches.
5. Consider the following assertion: "All the teams who played against a team that won all its matches won at least one match". Which of the following formulas formalize this assertion, and which are equivalent?
   (a) $\forall x \exists y\, (P(x,y) \wedge \forall z(P(y,z) \Rightarrow W(y,z)) \Rightarrow \exists v W(x,v))$.
   (b) $\forall x (\exists y\, (P(x,y) \wedge \forall z(P(y,z) \Rightarrow W(y,z))) \Rightarrow \exists v W(x,v))$.
   (c) $\exists x (\forall y\, (P(x,y) \Rightarrow W(x,y)) \Rightarrow \forall z(P(x,z) \Rightarrow \exists v W(x,v)))$.
   (d) $\forall x \forall y\, (P(x,y) \wedge \forall z(P(y,z) \Rightarrow W(y,z)) \Rightarrow \exists v W(x,v))$.
   (e) $\forall x (\forall y\, (P(x,y) \wedge \forall z(P(y,z) \Rightarrow W(y,z))) \Rightarrow \exists v W(x,v))$.

Exercise 67 (Formalization,)**
We define constants s for Serge, t for Tobby, $L(x,y)$ for x loves y, D(x) for x is a dog, $P(x)$ for x is a pet, $K(x)$ for x is a kid, $B(x)$ for x is a bird and $S(x,y)$ for x is scared of y. Describe the signature Σ associated for these symbols, and formalize the following statements in first order logic using that signature.

1. Dogs and birds are pets.
2. Tobby is a dog that loves kids.
3. Birds do not like dogs.
4. Serge loves all pets except dogs.
5. All kids are not scared of dogs.
6. Some dogs love kids.
7. Some dogs love kids, and vice versa.
8. Kids love some dogs.

Exercise 68 (Evaluation of unary predicates)
Let I be the interpretation on domain $D = \{0,1\}$ with $P_I = \{0\}$ and $Q_I = \{1\}$.

1. Evaluate the following formulas with I: $\forall x P(x)$ and $\forall x(P(x) \vee Q(x))$.
2. Are the formulas $\forall x P(x) \vee \forall x Q(x)$ and $\forall x(P(x) \vee Q(x))$ equivalent?
3. Evaluate the following formulas with I: $\exists x P(x)$ and $\exists x(P(x) \wedge Q(x))$.
4. Are the formulas $\exists x P(x) \wedge \exists x Q(x)$ and $\exists x(P(x) \wedge Q(x))$ equivalent?
5. Evaluate the following formulas with I: $\forall x(P(x) \Rightarrow Q(x))$ and $\forall x P(x) \Rightarrow \forall x Q(x)$.
6. Are the formulas $\forall x(P(x) \Rightarrow Q(x))$ and $\forall x P(x) \Rightarrow \forall x Q(x)$ equivalent?

Exercise 69 (Interpretation,*)
Consider the following formulas:

1. $\forall x \exists y (y = x + 1)$.
2. $\exists y \forall x (y = x + 1)$.
3. $\forall x \exists y (y = x + 1) \Rightarrow \exists y \forall x (y = x + 1)$.
4. $\forall x \exists y (x = y + 1)$.
5. $\exists x \forall y (y = x + y)$.
6. $\exists x (x \neq 0 \wedge x + x = x)$.

and the following interpretations:

1. I1 is Boolean Algebra on $\{0,1\}$.
2. I2 is the usual arithmetic on natural numbers.
3. I3 is the usual arithmetic on rational numbers.

4. I4 is the Boolean Algebra on $\mathcal{P}(X)$ where constants 0 and 1 denote the sets $\emptyset$ and X, and addition represents set union.

Indicate if these interpretations are models or counter-models of the formulas above.

Exercise 70 (Unary predicats and equality)
Let I be the interpretation on the domain $D = \{0, 1, 2\}$ with $P_I = \{0, 1\}, Q_I = \{1, 2\}, R_I = \{\}$. Evaluate the following formulas with this interpretation:

1. $\exists x R(x)$.
2. $\forall x(P(x) \vee Q(x))$.
3. $\forall x(P(x) \Rightarrow Q(x))$.
4. $\forall x(R(x) \Rightarrow Q(x))$.
5. $\exists x(P(x) \wedge \forall y(P(y) \Rightarrow x = y))$[4].
6. $\exists x(P(x) \wedge Q(x) \wedge \forall y(P(y) \wedge Q(y) \Rightarrow x = y))$.

Exercise 71 (Evaluation, equality)
We use the unary function symbol f and the constant a. Let $x \neq y$ be a shorthand for $\neg(x = y)$. We define interpretations I_1, I_2 on domain $\{0, 1, 2\}$ as follows:
$a_{I_1} = a_{I_2} = 0$.

x	$f_{I_1}(x)$	$f_{I_2}(x)$
0	0	1
1	0	2
2	2	0

In interpretation I_1, then in I_2, evaluate the following formula:

1. $f(a) = a$.
2. $f(f(a)) = a$.
3. $f(f(f(a)) = a$.
4. $\exists x(f(x) = x)$, (f has a fixed point).
5. $\forall x(f(f(f(x))) = x)$.
6. $\forall y \exists x(f(x) = y)$, ($f$ is surjective, or onto).
7. $\forall x \forall y(f(x) = f(y) \Rightarrow x = y)$, ($f$ is injective, or one-to-one.
8. $\neg \exists x \exists y(f(x) = f(y) \wedge x \neq y)$.

Exercise 72 (Formalization and evaluation)
We use the following notation:

- $P(x)$ means x passed his exam
- $Q(x, y)$ means x called y

Give the signature associated with these symbols, and formalize the following statements:

1. Someone failed the exam and was not called by anyone.
2. All those who passed the exam received a call.
3. Nobody called those who passed the exam.
4. All those cho called someone else called someone who passed the exam.

[4] This formula means that there is only one element satisfying P.

We define an interpretation with domain $D = \{0, 1, 2, 3\}$. Let Anatoli, Boris, Catarina and Denka be four constants with value 0, 1, 2, 3 respectively. Anatoli and Boris are boys, Catarina and Denka are girls. In our interpretation, only Boris and Catarina passed the exam, every boy called every girl, Denka called Boris, Catarina called Denka and no other calls were made. Evaluate the statements above in this interpretation.
Indication : to facilitate the evaluation, it might help to represent the interpretation pictorially, by circling people who passed the exam and drawing an arrow from x to y if x called y.

Exercise 73 (Expansion and counter-models)
Using the method of expansion, find counter-models to the following formulas:

1. $\exists x P(x) \Rightarrow \exists x(P(x) \wedge Q(x))$.
2. $\forall x(P(x) \Rightarrow Q(x)) \Rightarrow \exists x Q(x)$.
3. $\forall x(P(x) \Rightarrow Q(x)) \Rightarrow (\exists x P(x) \Rightarrow \forall x Q(x))$.
4. $(\exists x F(x) \Rightarrow \exists x G(x)) \Rightarrow \forall x(F(x) \Rightarrow G(x))$.
5. $\forall x \exists y R(x, y) \Rightarrow \exists x R(x, x)$.
6. $\forall x \forall y(R(x, y) \Rightarrow R(y, x)) \Rightarrow \forall x R(x, x)$.

Hint : it should be enough to build 1 or 2 expansions.

Exercise 74 (Incorrect reasoning)
Consider the following hypotheses:

1. $\exists x P(x)$.
2. $\exists x Q(x)$.
3. $\forall x(P(x) \wedge Q(x) \Rightarrow R(x))$.

Show that it is incorrect to deduce from these that $\exists x R(x)$.

Exercise 75 (Counter-model with a relation)
Construct counter-models for the following formulas, where F is a relation:

1. $\forall x \exists y(x = y) \Rightarrow \exists y \forall x(y = x)$.
2. $F(a) \wedge (a \neq b) \Rightarrow \neg F(b)$.
3. $\exists x \exists y(F(x) \wedge F(y) \wedge x \neq y) \Rightarrow \forall x F(x)$.
4. $\forall x \forall y(F(x, y) \Rightarrow x = y) \Rightarrow \exists x F(x, x)$.

Exercise 76 (Counter-model with a function)
Construct counter-models for the following formulas, where f is a function and P is a relation :

1. $\forall y \exists x(f(x) = y)$.
2. $\forall x \forall y(f(x) = f(y) \Rightarrow x = y)$.
3. $\exists x \forall y(f(y) = x)$.
4. $\forall x(P(x) \Rightarrow P(f(x)))$.

Exercise 77 (Equivalences)
Prove that

1. $\neg \forall x \exists y P(x, y) \equiv \exists y \forall x \neg P(y, x)$.
2. $\exists x(P(x) \Rightarrow Q(x)) \equiv \forall x P(x) \Rightarrow \exists x Q(x)$.

3. The sentence “no sick person likes charlatans” was formalized in first order logic by two students as follows:
    - $\forall x \forall y((S(x) \wedge L(x, y)) \Rightarrow \neg C(y))$.
    - $\neg(\exists x(S(x) \wedge (\exists y(L(x, y) \wedge C(y)))))$.

   Show that both students are saying the same thing, that is, these two formulas are equivalent.

Exercise 78 (✒ , Proofs,*)
Prove the following two equivalences from Lemma 4.4.1:

- $\neg \exists x A \equiv \forall x \neg A$.
- $\exists x A \equiv \neg \forall x \neg A$.

Hint : use properties 1 and 2 from Lemma 4.4.1.

Exercise 79 (Proof)
We know that $(\forall x(A \wedge B)) \equiv (A \wedge (\forall x B))$ under the condition that x is not a free variable in A, as mentioned in paragraph 4.4.2. Show that this condition is necessary by giving an assignment that give different values to the formulas $\forall x(P(x) \wedge Q(x))$ and $P(x) \wedge (\forall x Q(x))$ when this condition is not met.

Exercises for Chapter II.5

Exercise 80 (Herbrand Universe)
Let Σ be the signature with constant a and function symbols f and g of arity one and two respectively.

- Give 5 distinct elements of the Herbrand universe of this signature.
- Give an inductive definition of this Herbrand universe.

Exercise 81 (Signature, Herbrand Universe and Base)
For each of the following sets of formulas,

- $\Gamma_1 = \{P(x) \vee Q(x) \vee R(x), \neg P(a), \neg Q(b), \neg R(c)\}$.
- $\Gamma_2 = \{P(x), \neg Q(x), \neg P(f(x)) \vee Q(f(x))\}$.
- $\Gamma_3 = \{P(x), Q(f(x)), \neg R(f(f(x))), \neg P(f(f(x))) \vee \neg Q(x) \vee R(f(x))\}$.

1. Give the signature, and the corresponding Herbrand universe and base.
2. Prove whether or not their universal closure has a model.

Exercise 82 (Herbrand's method)
Use Herbrand's method in order to prove that the following set of formulae is unsatisfiable:

1. $\forall x R(x, f(x))$
2. $\forall x \forall y(\overline{S(x,y)} \vee R(x,y) \vee R(y,x))$
3. $\forall x \forall y(S(x,y) \vee \overline{R(x,y)})$
4. $\forall x \forall y(S(x,y) \vee \overline{R(y,x)})$
5. $\forall y \overline{S(y,a)}$

Exercise 83 (Herbrand's method,*)
Use Herbrand's method in order to prove that the following set of formulae is unsatisfiable:

1. $\forall x(Q(x) \vee \neg P(f(x)))$
2. $\forall y(Q(y) \Rightarrow R(y))$
3. $\forall z(\neg P(z) \Rightarrow Q(z) \vee R(z))$
4. $\forall u \neg R(u)$

In particular, you will have to transform your set of unsatisfiable instances into an equivalent set of clauses. Then, you will prove that it is contradictory with a proof in propositional resolution.

Exercise 84 (Herbrand's method,*)
Let Γ be the following set of formulae:

1. $\overline{S(x,y)} + \overline{M(z,x)} + M(z,y)$
2. $S(x,y) + M(f(x,y),x)$
3. $S(x,y) + \overline{M(f(x,y),y)}$
4. $S(c,a)$
5. $S(a,b)$
6. $\overline{S(c,b)}$

Find a finite unsatisfiable set of closed instances of these formulae.

This enables us to deduce something about a set of first-order formulae : which set and what ?

Exercise 85 (Herbrand Model,*)**
Let Δ be the following set of formulas:

1. $x < y \wedge y < z \Rightarrow x < z$.
2. $\neg(x < x)$.
3. $x < y \Rightarrow x < f(x,y) \wedge f(x,y) < y$.
4. $a < b$.

- Give a model of $\forall(\Delta)$.
- Does $\forall(\Delta)$ have a model on the Herbrand universe created from a, b, f ?

Exercise 86 (Skolemization)
Skolemize the following formulas (pay attention to the negations !) then transform them in clausal form.

1. $\neg(\exists x P(x) \vee \exists x Q(x) \Rightarrow \exists x(P(x) \vee Q(x)))$.
2. $\neg(\forall x \forall y \forall z(e(x,y) \wedge e(y,z) \Rightarrow \neg e(x,z)) \Rightarrow \neg\exists x \forall y\ e(x,y))$.
3. $\neg(\neg\forall x P(x) \vee \neg\forall x Q(x) \Rightarrow \neg(\forall x P(x) \wedge \forall x Q(x)))$.
4. $\forall x((\exists y P(x,y) \Rightarrow \exists x Q(x)) \wedge \exists y P(x,y) \wedge \neg\exists x Q(x))$.
5. $\neg(\exists x \forall y \forall z((P(y) \Rightarrow Q(z)) \Rightarrow (P(x) \Rightarrow Q(x))))$

Exercise 87 (Unification)
Are the following terms unifiable? If so, give their most general unifier, if not, justify your answer.

- $h(g(x), f(a,y), z)$ et $h(y, z, f(u,x))$.
- $h(g(x), f(a,y), z)$ et $h(y, z, f(u, g(x)))$.

Exercise 88 (Unification)
Give the most general unifier for each of the following terms if it exists.

1. $R(a, f(x)) = R(y, f(g(y, b)))$,
2. $R(y, f(x)) = R(x, f(g(y, b)))$,
3. $Q(y, f(a), f(x)) = Q(g(a, b), x, f(y))$, et
4. $Q(y, f(x), g(y, x)) = Q(x, f(y), g(f(a), y))$.

Exercise 89 (Unification)
Give the most general unifier for each of the following terms if it exists.

1. $pair(a, crypt(z, b))$ et $pair(x, y)$.
2. $pair(crypt(x, b), crypt(y, b))$ et $pair(crypt(a, b), z)$.
3. $crypt(pair(z, a), x)$ et $crypt(pair(y, crypt(x, b)), b)$.
4. $crypt(pair(a, z), x)$ et $crypt(pair(y, crypt(x, b)), b)$.
5. $f(x, y, g(a, a))$ et $f(g(y, y), z, z)$
6. $f(x, y, a)$ et $f(y, g(z, z), x)$

Exercise 90 (Unification with multiple solutions)
The equation $f(g(y), y) = f(u, z)$ has two most general unifiers (Recall: they are therefore equivalent). Give these two solutions.

Exercise 91 (Proof by resolution,*)
Show that the following formula is valid by transforming its *negation* in clausal form and by finding a contradictory set of instances of the clauses.

$$\forall x(P(x) \Rightarrow Q(x)) \Rightarrow (\forall x P(x) \Rightarrow \forall x Q(x)).$$

Exercise 92 (Proof by resolution,)**
Consider the following formulas:

1. $H_1 = \exists x P(x) \Rightarrow \forall x P(x)$.
2. $H_2 = \forall x(P(x) \vee Q(x))$.
3. $C = \exists x \neg Q(x) \Rightarrow \forall x P(x)$.

We want to show that C is the consequence of H_1 and H_2 using instantiation and resolution.

1. Transform the three formulas H_1, H_2, $\neg C$ in clausal form.
2. Find contradictory instances of the clauses obtained and show by propositional resolution that these instances are contradictory.
3. Give a direct proof of this contradiction using factorization, copy and binary resolution. It is possible that you only need the last rule.

Exercise 93 (Proof by resolution,)**

- Using a proof by factorization, copy and binary resolution, prove that the universal closure of the following set of clauses is unsatisfiable:

  1. $P(f(x)) \vee \neg Q(y, a)$.
  2. $Q(a, a) \vee R(x, x, b) \vee S(a, b)$.
  3. $S(a, z) \vee \neg R(x, x, b)$.
  4. $\neg P(f(c)) \vee R(x, a, b)$.
  5. $\neg S(y, z) \vee \neg S(a, b)$.

- Using a proof by factorization, copy and binary resolution, prove that the universal closure of the following set of clauses is unsatisfiable:

  1. $P(x)$.
  2. $\neg P(y) \vee Q(y, x)$.
  3. $\neg Q(x, a) \vee \neg Q(b, y) \vee \neg Q(b, a) \vee \neg P(f(y))$.

- Using a proof by factorization, copy and binary resolution, prove that the universal closure of the following set of clauses is unsatisfiable:

  1. $R(x, a)$.
  2. $R(y, b)$.
  3. $R(z, a + b)$.
  4. $\neg(b < x + b) \vee (a + b < a)$.
  5. $b < z$.
  6. $\neg(a + b < x) \vee \neg R(y, b)$.

  Recall : The symbol $+$ has a higher priority than the symbol $<$, which itself has a higher priority than binary connectors.

Exercise 94 (Unification, resolution)
Let $\Gamma_1 = \{P(x, f(x, b), u), \neg P(g(a), z, h(z))\}$ and $\Gamma_2 = \{P(x, f(x, b), u), \neg P(g(z), z, h(z))\}$
Are the sets $\forall(\Gamma_1)$ and $\forall(\Gamma_2)$ satisfiable or unsatisfiable?

Exercise 95 (Skolemization and First Order Resolution)
The goal of this exercise is to prove the following syllogism:

$$\forall x(man(x) \Rightarrow mortal(x)) \wedge man(Socrates) \Rightarrow mortal(Socrates)$$

- Skolemize the **negation** of the syllogism.
- Transform the obtained Skolem form in clausal form.
- Demonstrate by instantiation that the **negation** of the syllogism is a contradiction.
- Demonstrate using factorization, copy and binary resolution that the negation of the syllogism is a contradiction.

Exercise 96 (Clausal Form and Resolution,)**
Consider the following formulas:

1. $A_1 = \exists u \forall v (P(u) \wedge (R(v) \Rightarrow Q(u, v)))$.
2. $A_2 = \forall u \forall v (\neg P(u) \vee \neg S(v) \vee \neg Q(u, v))$.
3. $A_3 = \exists v (R(v) \wedge S(v))$.

Show that the set containing these three formulas is contradictory using resolution:

1. Transorm A_1, A_2, A_3 in clausal form.
2. Find contradictory instances of the clauses you obtained and show this contradiction using propositional resolution.
3. Make a direct proof of $\perp$ using factorization, copy and binary resolution.

Exercise 97 (✑ Coherence of First Order Resolution)
Prove the coherence of first order resolution, that is:

Let Γ be a set of clauses and C a clause. If $\Gamma \vdash_{1fcb} C$ then $\forall(\Gamma) \models \forall(C)$.

Exercise 98 (Clausal Form and Resolution,)**
Consider the following formulas:

1. $E_1 = \forall x (x = x)$.
2. $E_2 = \forall x \forall y (x = y \Rightarrow y = x)$.
3. $E_3 = \forall x \forall y \forall z \forall t (x = y \wedge z = t \Rightarrow (x \in z \Rightarrow y \in t))$.
4. $C = \forall y \forall z (y = z \Rightarrow \forall x (x \in y \Leftrightarrow x \in z))$.

E_1, E_2, E_3 are axioms of equality: E_1 says equality is reflexive, E_2 says it is symmetric, E_3 says it is a congruence relative to $\in$. We do not use transitivity in this exercise. C says that two sets are equal if they have the same elements. We note that in this exercise, the symbol '=' is not treated as usual, as the symbol with fixed meaning describing the identity relation. The restriction on the appearance of the equality symbol in formulas applies only when it is considered as a symbol with fixed meaning. Show that C is a consequence of the other axioms on equality:

1. Put the formulas E_1, E_2, E_3, $\neg C$ in clausal form.
2. Find contradictory instances of the clauses you obtained.
3. Show a direct proof of $\perp$ using factorization, copy and binary resolution.

Exercise 99 (Clausal Form and Resolution,)**
Consider the following formulas:

1. $H_1 = \forall u (\exists v R(u, v) \Rightarrow R(u, f(u)))$.
2. $H_2 = \forall u \exists v R(u, v)$.
3. $H_3 = \exists u R(f(f(u)), u)$.
4. $C = \exists u \exists v \exists w (R(u, v) \wedge R(v, w) \wedge R(w, u))$.

Show that C is the consequence of H_1, H_2, H_3.

1. Transform the formulas H_1, H_2, H_3, $\neg C$ in clausal form.
2. Find contradictory instances of the clauses you obtained, and show the contradiction using propositional resolution.
3. Prove this contradiction directly using factorization, copy and binary resolution.

Exercise 100 (Clausal Form and Resolution,)**
Let $F(x, y)$ be a formalization of " x is y's father ". Let $A(x, y)$ be a formalization of " x is y's ancestor ". Consider the following formulas:

1. $H_1 = \forall x \exists y F(x, y)$.
2. $H_2 = \forall x \forall y \forall z (F(x, y) \wedge F(y, z) \Rightarrow A(x, z))$.
3. $C = \forall x \exists y A(x, y)$.

Show that C is the consequence of H_1, H_2 by transforming $H_1, H_2, \neg C$ into clausal form, and deriving the empty clause using factorization, copy and binary resolution.

Exercise 101 (Bernays-Schonfinkel Formula,*)**
A BS (Bernays-Schonfinkel) formulais a closed formula *without* any function symbols, of the form $\exists x_1 \ldots \exists x_n \forall y_1 \ldots \forall y_p \; B$, where B does not have any quantifiers. Show how to decide the satisfiability of such a formula.

Exercises for Chapter II.6

Exercise 102 (Natural Deduction)
Prove the following formulas using natural deduction:

1. The famous syllogism, "All men are mortal, Socrates is a man, therefore Socrates is mortal", which we formalize as $\forall x(H(x) \Rightarrow M(x)) \wedge H(socrates) \Rightarrow M(socrates)$.
2. $\forall x P(x) \Rightarrow \exists y P(y)$.
3. $\forall x P(x) \Rightarrow \exists x P(x)$.
4. $\forall x(P(x) \wedge Q(x)) \Rightarrow \forall x P(x) \wedge \forall x Q(x)$. Note that example 6.1.2 gives the proof of the reciprocal.
5. $\exists x(P(x) \vee Q(x)) \Rightarrow \exists x P(x) \vee \exists x Q(x)$.
6. $\forall x(P(x) \Rightarrow Q(x)) \wedge \exists x P(x) \Rightarrow \exists x Q(x)$.
7. $\exists x P(x) \vee \exists x Q(x) \Rightarrow \exists x(P(x) \vee Q(x))$. (**)
8. $\forall x(P(x) \Rightarrow Q(x)) \wedge \forall x(Q(x) \Rightarrow R(x)) \Rightarrow \forall x(P(x) \Rightarrow R(x))$.
9. $\forall x\ (P(x) \Rightarrow Q(x)) \wedge \exists x\ \neg Q(x) \Rightarrow \exists x\ \neg P(x)$.

In this exercise, the formulas $P(x)$ and $Q(x)$ can be replaced by *any* other formulas.

Exercise 103 (Natural Deduction)
Prove the following formulas:

1. $\forall x \forall y P(x, y) \Rightarrow \forall y \forall x P(x, y)$.
2. $\exists x \exists y P(x, y) \Rightarrow \exists y \exists x P(x, y)$.
3. $\exists x \forall y P(x, y) \Rightarrow \forall y \exists x P(x, y)$.
4. $\forall x(Q(x) \Rightarrow \forall y(R(y) \Rightarrow P(x, y))) \Rightarrow \forall y(R(y) \Rightarrow \forall x(Q(x) \Rightarrow P(x, y)))$. (*)

In this exercise, the formula $P(x, y)$ can be replaced by any other formula. However, $Q(x)$ can only be replaced by a formula that does not have y as a free variable, and $R(y)$ can only be replaced by a formula that does not have x as a free variable. Explain the reason for these constraints.

Exercise 104 (Find the Mistake)
Consider the following formula:

$$\exists x\ P(x) \wedge \forall x\ Q(x) \Rightarrow \exists x\ (P(x) \wedge Q(x)).$$

Among the following three natural deduction proofs, only one is correct. Identify the correct proof, and justify why the other two are incorrect.

1.

context	number	proof	justification
1	1	`Suppose` $\exists x\ P(x) \wedge \forall x\ Q(x)$	
1	2	$\exists x\ P(x)$	$\wedge$E1 1
1	3	$\forall x\ Q(x)$	$\wedge$E2 1
1,4	4	`Suppose` $P(x)$	
1,4	5	$Q(x)$	$\forall$E 3, x
1,4	6	$P(x) \wedge Q(x)$	$\wedge$I 4,5
1,4	7	$\exists x(P(x) \wedge Q(x))$	$\exists$I 6, x
1	8	`Thus` $P(x) \Rightarrow \exists x(P(x) \wedge Q(x))$	$\Rightarrow$I 4,7
1	9	$\exists x(P(x) \wedge Q(x))$	$\exists$E 2,8
	10	`Thus` $\exists x\ P(x) \wedge \forall x\ Q(x) \Rightarrow \exists x(P(x) \wedge Q(x))$	$\Rightarrow$I 1,9

2.

context	number	proof	justification
1	1	`Suppose` $\exists x\ P(x) \wedge \forall x\ Q(x)$	
1	2	$\exists x\ P(x)$	$\wedge$E1 1
1	3	$\forall x\ Q(x)$	$\wedge$E2 1
1,4	4	`Suppose` $P(x)$	
1,4	5	$Q(x)$	$\forall$E 3, x
1,4	6	$P(x) \wedge Q(x)$	$\wedge$I 4,5
1	7	`Thus` $P(x) \Rightarrow P(x) \wedge Q(x)$	$\Rightarrow$I 4,6
1	8	$P(x) \wedge Q(x)$	$\exists$E 2,7
1	9	$\exists x(P(x) \wedge Q(x))$	$\exists$I 9, x
	10	`Thus` $\exists x\ P(x) \wedge \forall x\ Q(x) \Rightarrow \exists x(P(x) \wedge Q(x))$	$\Rightarrow$I 1,9

3.

context	number	proof	justification
1	1	`Suppose` $\exists x\ P(x) \wedge \forall x\ Q(x)$	
1	2	$\exists x\ P(x)$	$\wedge$E1 1
1	3	$\forall x\ Q(x)$	$\wedge$E2 1
1,4	4	`Suppose` $P(x)$	
1	5	`Thus` $P(x) \Rightarrow P(x)$	$\Rightarrow$I 4,4
1	6	$P(x)$	$\exists$E 2,6
1	7	$Q(x)$	$\forall$E 3, x
1	8	$P(x) \wedge Q(x)$	$\wedge$I 7,8
1	9	$\exists x(P(x) \wedge Q(x))$	$\exists$I 9, x
	10	`Thus` $\exists x\ P(x) \wedge \forall x\ Q(x) \Rightarrow \exists x(P(x) \wedge Q(x))$	$\Rightarrow$I 1,10

Exercise 105 (Copy)
Prove $\forall x \forall y P(x, y) \Rightarrow \forall x \forall y P(y, x)$ using natural deduction with as few copies as possible.

Exercise 106 (Natural Deduction)
Prove the following formulas using natural deduction (note that Q is a propositional variable):

1. $\forall x(Q \wedge P(x)) \Rightarrow Q \wedge \forall x P(x)$.
2. $Q \wedge \forall x P(x) \Rightarrow \forall x(Q \wedge P(x))$.
3. $\forall x(Q \vee P(x)) \Rightarrow Q \vee \forall x P(x)$. (**)
4. $Q \vee \forall x P(x) \Rightarrow \forall x(Q \vee P(x))$.
5. $\exists x(Q \wedge P(x)) \Rightarrow Q \wedge \exists x P(x)$.
6. $Q \wedge \exists x P(x) \Rightarrow \exists x(Q \wedge P(x))$.
7. $\exists x(Q \vee P(x)) \Rightarrow Q \vee \exists x P(x)$.
8. $Q \vee \exists x P(x) \Rightarrow \exists x(Q \vee P(x))$. (*)

In this exercise, the formula $P(x)$ can be replaced by any formula. However, Q can be replaced only by a formula that does not have x as a free variable.

Exercise 107 (Proof)
Prove the formula $\neg\exists x P(x) \Rightarrow \forall x \neg P(x)$. Verify that $P(x)$ can be replaced by any formula. Note: the reciprocal to this formula is proven in example 6.1.6.

Exercise 108 (Equality)
Prove the following formulae:

1. $R(a, c) \wedge (a = b) \Rightarrow R(b, c)$.
2. $x = y \Rightarrow f(x, z) = f(y, z)$.
3. $\forall x \exists y(x = y)$.
4. $\exists x \forall y\ x = y \Rightarrow \forall x \forall y\ x = y$. (*)

Exercise 109 (Equality,)**
Prove that the second definition of " there exists one and only one" (see subsection 6.2.3) implies the first, that is, prove the formula: $\exists x P(x) \wedge \forall x \forall y(P(x) \wedge P(y) \Rightarrow x = y) \Rightarrow \exists x(P(x) \wedge \forall y(P(y) \Rightarrow x = y))$.

Exercise 110 (Induction and natural deduction,*)**
We can define addition using the following formulas:

(a) $\forall n(n + 0 = n)$.

(b) $\forall n \forall p(n + s(p) = s(n + p))$.

These two formula allow us to do additions: we can prove, among other, that $s(0) + s(0) = s(s(0))$. But they do not allow the proof of more general properties of addition, which require the recurrence principle.

1. Show that from hypotheses (a) and (b), we cannot deduce $\forall n(0 + n = n)$.
2. We give the name $P(n)$ to the property above and describe the recurrence principle on this property as follows:

(c) $\forall n(P(n) \Leftrightarrow 0+n=n)$.

(d) $P(0) \wedge \forall n(P(n) \Rightarrow P(s(n))) \Rightarrow \forall n P(n)$.

Prove using natural deduction that $(a) \wedge (b) \wedge (c) \wedge (d) \Rightarrow \forall n P(n)$.

Exercise 111 (Exam 2009)
All of the following proofs must be justified.

1. Prove using natural deduction that this formula is valid:
$(\exists x\ p(x) \Rightarrow \forall x\ q(x)) \Rightarrow \forall x(p(x) \Rightarrow q(x))$.

2. Prove using natural deduction that this formula is valid:
$\exists x\ p(x) \wedge \forall x(p(x) \Rightarrow p(f(x))) \Rightarrow \exists x\ p(f(f(x)))$.

3. We denote by $f^n(x)$ the term obtained applying f n times to x.
For instance, $f^0(x) = x, f^1(x) = f(x), f^2(x) = f(f(x))$.
Let Γ, Δ be two formula sets and A, B two formulas. We remind $\Gamma \vdash A$ is true if there is a proof of A in the *environment* Γ.
We give some trivial properties of the $\vdash$ relation

$$\begin{aligned} \text{Monotony} &: \text{ if } \Gamma \vdash A \text{ and } \Gamma \subset \Delta \text{ then } \Delta \vdash A \\ \text{Composition} &: \text{ if } \Gamma \vdash A \text{ and } \Gamma \vdash A \Rightarrow B \text{ then } \Gamma \vdash B \end{aligned}$$

(a) Prove using natural deduction of the following property:
$\forall x(p(x) \Rightarrow p(f(x))) \vdash \exists x p(f^n(x)) \Rightarrow \exists x\ p(f^{n+1}(x))$

(b) Deduce from the above property, monotony and composition that *for any* natural integer n :
$\exists x\ p(x), \forall x(p(x) \Rightarrow p(f(x))) \vdash \exists x\ p(f^n(x))$.

Exercise 112 (Exam 2012)
Prove the following formulas using first-order natural deduction:

1. $\neg\forall x P(x) \vee \neg\exists y Q(y) \Rightarrow \neg(\forall x P(x) \wedge \exists y Q(y))$

2. $\forall x \forall y(P(y) \Rightarrow R(x)) \Rightarrow \exists y P(y) \Rightarrow \forall x R(x)$

3. $\neg\forall x \neg P(x) \Rightarrow \exists x P(x)$

Exercise 113 (Exam 2013)
Prove the following formulae using first order natural deduction.

1. $\exists x(Q(x) \Rightarrow P(x)) \wedge \forall x Q(x) \Rightarrow \exists x P(x)$

2. $\forall x\ \forall y(R(x,y) \Rightarrow \neg R(y,x)) \Rightarrow \forall x \neg R(x,x)$

Exercise 114 (Questions from various exams)
Prove the following formulae using first order natural deduction.

1. $\exists x(P(x) \vee Q(x)) \wedge \forall x \neg Q(x) \Rightarrow \exists x P(x)$

2. $\forall x(P(x) \Rightarrow Q(x)) \wedge \exists x(P(x) \wedge R(x)) \Rightarrow \exists x(Q(x) \wedge R(x))$

3. $\exists x \neg(P(x) \vee \neg P(x)) \Rightarrow \forall x P(x)$