Natural Deduction: quantifiers, copy and equality

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Motivation

Propositional case

There are algorithms to decide whether a given formula is valid or not.

First-order case

There is no algorithm to decide whether a given formula is valid or not.
## Motivation

### Propositional case

There are algorithms to decide whether a given formula is valid or not.

### First-order case

There is no algorithm to decide whether a given formula is valid or not.

If we assume the equivalence between provable and valid, there is no algorithm that, given a first-order formula, could:

- build a proof
- or warn us that this formula has no proof.

(as proved independently by Church in 1936 and Turing in 1937)
Overview

Introduction

Rules and examples

Copy rule

Rules for equality

Conclusion
Overview

Introduction

Rules and examples

Copy rule

Rules for equality

Conclusion
## Table 3.1

<table>
<thead>
<tr>
<th>Introduction</th>
<th>Elimination</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[A]$</td>
<td>$A \quad A \Rightarrow B$</td>
</tr>
<tr>
<td>$\ldots$</td>
<td>$\Rightarrow E$</td>
</tr>
<tr>
<td>$B$</td>
<td>$A \quad B$</td>
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<tr>
<td>$\frac{A \Rightarrow B}{\Rightarrow I}$</td>
<td>$\frac{A \Rightarrow B}{B}$</td>
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<tr>
<td>$A \quad B$</td>
<td>$\frac{A \land B}{\land I}$</td>
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<tr>
<td>$A \quad B$</td>
<td>$\frac{A \land B}{A \Rightarrow C \quad B \Rightarrow C}$</td>
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<tr>
<td>$\frac{A \land B}{\land E 1}$</td>
<td>$\frac{A \land B}{\land E 2}$</td>
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<tr>
<td>$A$</td>
<td>$\frac{A \lor B}{\lor I 1}$</td>
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<tr>
<td>$\frac{A \lor B}{\lor I 2}$</td>
<td>$\frac{A \lor B}{\lor E}$</td>
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<tr>
<td>$A$</td>
<td>$\frac{A \lor B}{\lor E}$</td>
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<tr>
<td>$B \lor A$</td>
<td>$\frac{A \lor B}{\lor E}$</td>
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<tr>
<td>Ex falso quodlibet</td>
<td>Ex falso quodlibet</td>
</tr>
<tr>
<td>$\bot$</td>
<td>$\frac{\bot}{\bot}$</td>
</tr>
<tr>
<td>$\frac{\bot}{\Rightarrow Efq}$</td>
<td>$\frac{\bot}{\Rightarrow Efq}$</td>
</tr>
<tr>
<td>Reductio ad absurdo</td>
<td>Reductio ad absurdo</td>
</tr>
<tr>
<td>$\neg \neg A$</td>
<td>$\frac{\neg \neg A}{A}$</td>
</tr>
<tr>
<td>$A$</td>
<td>$\frac{A}{\Rightarrow RAA}$</td>
</tr>
</tbody>
</table>
An extension of propositional natural deduction

- The definitions for **proof sketch**, **environment**, **context**, **usable formula** remain **the same**!
Introduction

An extension of propositional natural deduction

- The definitions for **proof sketch, environment, context, usable formula** remain the same!
- Still only one rule to remove hypotheses: $\Rightarrow I$. 
An extension of propositional natural deduction

- The definitions for **proof sketch**, **environment**, **context**, **usable formula** remain the same!
- Still only one rule to remove hypotheses: \( \Rightarrow I \).

Additional rules about

- quantifiers
- copy
- equality
Consistency and completeness
Consistency and completeness

- **Consistency**: $\Gamma \vdash A$ implies $\Gamma \models A$.

  Proved in the next lecture.
  The main point is to prove that the new rules are consistent.
Consistency and completeness

- **Consistency**: $\Gamma \vdash A$ implies $\Gamma \models A$.
  
  Proved in the next lecture.
  
  The main point is to prove that the new rules are consistent.

- **Completeness**: $\Gamma \models A$ implies $\Gamma \vdash A$.
  
  Assumed without proof.
Overview

Introduction

Rules and examples

Copy rule

Rules for equality

Conclusion
Quantifier rules

An elimination rule and an introduction rule for each quantifier.

- How to use these rules on examples.
- And some mistakes you can make if you don’t comply with the use conditions of these rules.
Reminder

Definition 4.3.34

Let \( x \) be a variable, \( t \) a term and \( A \) a formula.

1. \( A < x := t > \) is the formula obtained by replacing in \( A \) every free occurrence of \( x \) with the term \( t \).

2. The term \( t \) is free for \( x \) in \( A \) if the variables of \( t \) are not bound in the free occurrences of \( x \).
Reminder

Definition 4.3.34

Let $x$ be a variable, $t$ a term and $A$ a formula.

1. $A < x := t >$ is the formula obtained by replacing in $A$ every free occurrence of $x$ with the term $t$.
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Example

\[ A = \forall y P(x, y) \]

- Is $z$ free for $x$ in $A$?
Reminder

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$A = \forall y P(x, y)$

- Is $z$ free for $x$ in $A$? yes
- Is $g(y)$ free for $x$ in $A$?
Reminder

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$$A = \forall y P(x, y)$$

- Is $z$ free for $x$ in $A$? yes
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- Is $f(x)$ free for $y$ in $A$?
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- Is $z$ free for $x$ in $A$? yes
- Is $g(y)$ free for $x$ in $A$? no
- Is $f(x)$ free for $y$ in $A$? yes
Quantifier rules: $\forall E$

$A$ and $B$ are formulae, $x$ is a variable, $t$ is a term

$\forall$ Elimination

$\frac{\forall x A}{A < x := t >}$

$t$ must be free for $x$ in $A$. 
Example 6.1.1

Wrong use of the rule $\forall E$: where is the mistake?

<table>
<thead>
<tr>
<th>Line</th>
<th>Description</th>
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<tbody>
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<td>2</td>
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Example 6.1.1

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</tr>
<tr>
<td>2</td>
<td>$\exists y P(y, y)$ $\forall E$ 1, $y$ ERROR</td>
</tr>
<tr>
<td>3</td>
<td>Therefore $\forall x \exists y P(x, y) \Rightarrow \exists y P(y, y)$</td>
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On line 2, the use conditions of $\forall E$ are not met because the term $y$ isn’t free for $x$ in the formula $\exists y P(x, y)$. 
Example 6.1.1

**Wrong use of the rule ∀E: where is the mistake?**

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<td>Assume ( \forall x \exists y P(x, y) )</td>
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<td>( \exists y P(y, y) )</td>
<td>( \forall E ) 1, ( y ) ERROR</td>
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On line 2, the use conditions of \( \forall E \) are not met because the term \( y \) isn’t free for \( x \) in the formula \( \exists y P(x, y) \).

Let \( I \) be the interpretation with domain \{0, 1\} such that \( P_I = \{(0, 1), (1, 0)\} \)

This interpretation makes the “conclusion” false.
Quantifier rules: $\forall$ I

$A$ and $B$ are formulae, $x$ is a variable.

$\forall$ Introduction

$$\frac{A}{\forall x A} \forall I$$

$x$ must be free
- neither in the *environment* of the proof,
- nor in the *context* of the premise of the rule.
Example 6.1.2 $\forall y P(y) \land \forall y Q(y) \Rightarrow \forall x (P(x) \land Q(x))$
Example 6.1.2 $\forall y P(y) \land \forall y Q(y) \Rightarrow \forall x (P(x) \land Q(x))$

1 1 Assume $\forall y P(y) \land \forall y Q(y)$

Remark: When using rule $\forall E$ on lines 4 and 5, we specify that $y$ has been replaced with $x$. 
Example 6.1.2 \( \forall y P(y) \land \forall y Q(y) \Rightarrow \forall x (P(x) \land Q(x)) \)

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<td>( \forall y P(y) )</td>
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</tr>
<tr>
<td>1</td>
<td>2</td>
<td>( \forall y P(y) )  ( \land E1 \ 1 )</td>
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<tr>
<td>1</td>
<td>3</td>
<td>( \forall y Q(y) )  ( \land E2 \ 1 )</td>
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Example 6.1.2 $\forall y P(y) \land \forall y Q(y) \Rightarrow \forall x (P(x) \land Q(x))$

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<tr>
<td>1</td>
<td>2</td>
<td>$\forall y P(y)$ $\land E$ 1 1</td>
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<tr>
<td>1</td>
<td>3</td>
<td>$\forall y Q(y)$ $\land E$ 2 1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>$P(x)$ $\forall E$ 2, $x$</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>$Q(x)$ $\forall E$ 3, $x$</td>
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**Remark**: When using rule $\forall E$ on lines 4 and 5, we specify that $y$ has been replaced with $x$. 
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<td>Assume $\forall y P(y) \land \forall y Q(y)$</td>
<td>$\land E1$ 1</td>
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<tr>
<td>1</td>
<td>2</td>
<td>$\forall y P(y)$</td>
<td>$\land E1$ 1</td>
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<td>3</td>
<td>$\forall y Q(y)$</td>
<td>$\land E2$ 1</td>
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<td>1</td>
<td>4</td>
<td>$P(x)$</td>
<td>$\forall E 2, x$</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>$Q(x)$</td>
<td>$\forall E 3, x$</td>
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<tr>
<td>1</td>
<td>6</td>
<td>$P(x) \land Q(x)$</td>
<td>$\land I$ 4, 5</td>
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**Remark:** When using rule $\forall E$ on lines 4 and 5, we specify that $y$ has been replaced with $x$. 

Example 6.1.2 $\forall y P(y) \land \forall y Q(y) \Rightarrow \forall x (P(x) \land Q(x))$

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<td>$\forall y P(y)$</td>
</tr>
<tr>
<td>1 3</td>
<td>$\forall y Q(y)$</td>
</tr>
<tr>
<td>1 4</td>
<td>$P(x)$</td>
</tr>
<tr>
<td>1 5</td>
<td>$Q(x)$</td>
</tr>
<tr>
<td>1 6</td>
<td>$P(x) \land Q(x)$</td>
</tr>
<tr>
<td>1 7</td>
<td>$\forall x (P(x) \land Q(x))$</td>
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Remark: When using rule $\forall E$ on lines 4 and 5, we specify that $y$ has been replaced with $x$. 
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<td>5</td>
<td>$Q(x)$</td>
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<tr>
<td>1</td>
<td>6</td>
<td>$P(x) \land Q(x)$</td>
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<tr>
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<td>7</td>
<td>$\forall x (P(x) \land Q(x))$</td>
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<tr>
<td>8</td>
<td></td>
<td>Therefore $\forall y P(y) \land \forall y Q(y) \Rightarrow \forall x (P(x) \land Q(x))$</td>
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**Remark**: When using rule $\forall E$ on lines 4 and 5, we specify that $y$ has been replaced with $x$. 
Example 6.1.3

Wrong use of the rule $\forall I$

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<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>Assume $P(x)$</td>
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<tr>
<td>1</td>
<td>2</td>
<td>$\forall x P(x)$ $\forall I$ 1</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>Therefore $P(x) \Rightarrow \forall x P(x)$ $\Rightarrow I$ 1, 2</td>
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Example 6.1.3

Wrong use of the rule $\forall I$

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<td>1</td>
<td>Assume $P(x)$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$\forall x P(x)$       $\forall I$ 1 ERROR</td>
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<tr>
<td>3</td>
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<td>Therefore $P(x) \Rightarrow \forall x P(x)$ $\Rightarrow I$ 1, 2</td>
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On line 2, $x$ is free in the context $P(x)$, which disallows generalisation on $x$. 
Example 6.1.3

Wrong use of the rule $\forall I$

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On line 2, $x$ is free in the context $P(x)$, which disallows generalisation on $x$.

Let $I$ be the interpretation with domain $\{0, 1\}$ such that $P_I = \{0\}$.
Let $e$ be a state where $x = 0$.
The assignment $(I, e)$ makes the “conclusion” false.
Quantifier rules: $\exists E$

$A$ and $B$ are formulae, $x$ is a variable.

$\exists E$ Elimination

\[
\exists x A \quad (A \Rightarrow B) \quad \exists E
\]

$x$ must be free

- neither in the environment,
- nor in $B$,
- nor in the context of $A \Rightarrow B$. 

Example 6.1.4

Wrong use of the rule $\exists E$

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<td>1</td>
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Wrong use of the rule $\exists E$

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<td>1 2</td>
<td>$\exists x P(x)$</td>
</tr>
<tr>
<td>1 3</td>
<td>$P(x) \Rightarrow \forall y Q(y)$</td>
</tr>
<tr>
<td>1 4</td>
<td>$\forall y Q(y)$</td>
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<tr>
<td>5</td>
<td>Therefore $\exists x P(x) \land (P(x) \Rightarrow \forall y Q(y)) \Rightarrow \forall y Q(y)$</td>
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The context of the premise $P(x) \Rightarrow \forall y Q(y)$ must not depend on $x$. 
Example 6.1.4

Wrong use of the rule $\exists E$

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The context of the premise $P(x) \Rightarrow \forall y Q(y)$ must not depend on $x$.

Let $I$ be the interpretation with domain $\{0, 1\}$ such that $P_I = Q_I = \{0\}$. Let $e$ be the state where $x = 1$. The assignment $(I, e)$ makes this “conclusion” false.
Example 6.1.5

Wrong use of the rule $\exists E$

1  1  Assume $\exists x P(x)$
1, 2 2  Assume $P(x)$
1 3  Therefore $P(x) \Rightarrow P(x)$  $\Rightarrow I\ 2,\ 2$
1 4  $P(x)$  $\exists E\ 1,\ 3$
1 5  $\forall x P(x)$  $\forall I\ 4$
6  Therefore $\exists x P(x) \Rightarrow \forall x P(x)$
Example 6.1.5

Wrong use of the rule $\exists E$

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<td>1</td>
<td>Assume $\exists x P(x)$</td>
</tr>
<tr>
<td>1, 2</td>
<td>2</td>
<td>Assume $P(x)$</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>Therefore $P(x) \Rightarrow P(x)$ $\Rightarrow I \ 2, \ 2$</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>$P(x)$ $\exists E \ 1, \ 3$ ERROR</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>$\forall x P(x)$ $\forall I \ 4$</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>Therefore $\exists x P(x) \Rightarrow \forall x P(x)$</td>
</tr>
</tbody>
</table>

The conclusion of rule $\exists E$ must not depend on $x$. 
Example 6.1.5

Wrong use of the rule $\exists E$

<table>
<thead>
<tr>
<th>Step</th>
<th>Assumption</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\exists x P(x)$</td>
<td>$\exists x P(x)$</td>
</tr>
<tr>
<td>1, 2</td>
<td>$P(x)$</td>
<td>$\exists E$ 1, 3 ERROR</td>
</tr>
<tr>
<td>1</td>
<td>$P(x) \Rightarrow P(x)$</td>
<td>$\exists E$ 1, 3 ERROR</td>
</tr>
<tr>
<td>1</td>
<td>$\forall x P(x)$</td>
<td>$\forall I$ 4</td>
</tr>
<tr>
<td>6</td>
<td>Therefore $\exists x P(x) \Rightarrow \forall x P(x)$</td>
<td></td>
</tr>
</tbody>
</table>

The conclusion of rule $\exists E$ must not depend on $x$.

Let $I$ be the interpretation with domain $\{0, 1\}$ such that $P_I = \{0\}$. $I$ make the “conclusion” false.
Quantifier rules: $\exists I$

$A$ and $B$ are formulae, $x$ is a variable, $t$ is a term

\[ A \prec x := t > \exists I \]

$\exists x A$

$t$ must be free for $x$ in $A$. 
Example 6.1.6 \( \neg \forall x A \Rightarrow \exists x \neg A \) (De Morgan’s law)

\[ \begin{array}{c}
1. \neg \forall x A \\
2. \neg \exists x \neg A \\
3. \neg \neg A \\
4. \exists x \neg A \\
5. \bot \\
6. \neg \neg \exists x \neg A \\
7. A \\
8. \forall x A \\
9. \bot
\end{array} \]

On line 4: we use \( \neg A = \neg A \langle x \rangle = \neg A \rangle \) and a variable \( x \) is always free for itself in \( A \).
Example 6.1.6 $\neg \forall x A \Rightarrow \exists x \neg A$ (De Morgan’s law)

1. Assume $\neg \forall x A$

On line 4: we use $\neg A = \neg A \neq x := x >$ and a variable $x$ is always free for itself in $A$.  

$\forall x A$
Example 6.1.6  \( \neg \forall x A \Rightarrow \exists x \neg A \) (De Morgan’s law)

1  1  Assume \( \neg \forall x A \)
1, 2  2  Assume \( \neg \exists x \neg A \)
Example 6.1.6 $\neg\forall x A \Rightarrow \exists x \neg A$ (De Morgan’s law)

1 1 Assume $\neg\forall x A$
1, 2 2 Assume $\neg\exists x \neg A$
1, 2, 3 3 Assume $\neg A$
Example 6.1.6 $\neg \forall x A \Rightarrow \exists x \neg A$ (De Morgan’s law)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>Assumption</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$\neg \forall x A$</td>
</tr>
<tr>
<td>1, 2</td>
<td>2</td>
<td>$\neg \exists x \neg A$</td>
</tr>
<tr>
<td>1, 2, 3</td>
<td>3</td>
<td>$\neg A$</td>
</tr>
<tr>
<td>1, 2, 3</td>
<td>4</td>
<td>$\exists x \neg A$</td>
</tr>
</tbody>
</table>

$\exists I/3, x$

- On line 4: we use $\neg A = \neg A < x := x >$
- and a variable $x$ is always free for itself in $A$. 
Example 6.1.6 $\neg \forall x A \Rightarrow \exists x \neg A$ (De Morgan’s law)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>1</th>
<th>Assume $\neg \forall x A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 2</td>
<td>2</td>
<td>Assume $\neg \exists x \neg A$</td>
<td></td>
</tr>
<tr>
<td>1, 2, 3</td>
<td>3</td>
<td>Assume $\neg A$</td>
<td></td>
</tr>
<tr>
<td>1, 2, 3</td>
<td>4</td>
<td>$\exists x \neg A$</td>
<td>$\exists I$ 3, $x$</td>
</tr>
<tr>
<td>1, 2, 3</td>
<td>5</td>
<td>$\bot$</td>
<td>$\Rightarrow E$ 2, 4</td>
</tr>
</tbody>
</table>

- On line 4: we use $\neg A = \neg A < x := x >$
  and a variable $x$ is always free for itself in $A$. 
Example 6.1.6 $\neg \forall x A \Rightarrow \exists x \neg A$ (De Morgan’s law)

1. Assume $\neg \forall x A$

2. Assume $\neg \exists x \neg A$

3. Assume $\neg A$

4. $\exists x \neg A$  \hspace{1cm} $\exists I$ 3, $x$

5. $\bot$  \hspace{1cm} $\Rightarrow E$ 2, 4

6. Therefore $\neg \neg A$  \hspace{1cm} $\Rightarrow I$ 3, 5

On line 4: we use $\neg A = \neg A < x := x >$
and a variable $x$ is always free for itself in $A$. 
Example 6.1.6 $\neg\forall x A \Rightarrow \exists x \neg A$ (De Morgan’s law)

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>Assume $\neg\forall x A$</td>
</tr>
<tr>
<td>1, 2</td>
<td>2</td>
<td>Assume $\neg\exists x \neg A$</td>
</tr>
<tr>
<td>1, 2, 3</td>
<td>3</td>
<td>Assume $\neg A$</td>
</tr>
<tr>
<td>1, 2, 3</td>
<td>4</td>
<td>$\exists x \neg A$</td>
</tr>
<tr>
<td>1, 2, 3</td>
<td>5</td>
<td>$\bot$</td>
</tr>
<tr>
<td>1, 2</td>
<td>6</td>
<td>Therefore $\neg \neg A$</td>
</tr>
<tr>
<td>1, 2</td>
<td>7</td>
<td>$A$</td>
</tr>
</tbody>
</table>

- On line 4: we use $\neg A = \neg A < x := x >$ and a variable $x$ is always free for itself in $A$. 
Example 6.1.6 \( \neg \forall x A \Rightarrow \exists x \neg A \) (De Morgan’s law)

1. Assume \( \neg \forall x A \)
2. Assume \( \neg \exists x \neg A \)
3. Assume \( \neg A \)
4. \( \exists x \neg A \) \( \exists I \) \( 3, x \)
5. \( \bot \) \( \Rightarrow E \) \( 2, 4 \)
6. Therefore \( \neg \neg A \) \( \Rightarrow I \) \( 3, 5 \)
7. \( A \) \( \text{Raa 6} \)
8. \( \forall x A \) \( \forall I \) \( 7 \)

On line 4: we use \( \neg A = \neg A < x := x > \) and a variable \( x \) is always free for itself in \( A \).
Example 6.1.6 \( \neg \forall x A \Rightarrow \exists x \neg A \) (De Morgan’s law)

1. Assume \( \neg \forall x A \)
2. Assume \( \neg \exists x \neg A \)
3. Assume \( \neg A \)
4. \( \exists x \neg A \) \( \exists I \ 3, x \)
5. \( \bot \) \( \Rightarrow E \ 2, 4 \)
6. Therefore \( \neg \neg A \) \( \Rightarrow I \ 3, 5 \)
7. \( A \) \( \text{Raa 6} \)
8. \( \forall x A \) \( \forall I \ 7 \)
9. \( \bot \) \( \Rightarrow E \ 1, 8 \)

▶ On line 4: we use \( \neg A = \neg A < x := x > \)
and a variable \( x \) is always free for itself in \( A \).
Example 6.1.6  \( \neg \forall x A \Rightarrow \exists x \neg A \) (De Morgan’s law)

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>Assume ( \neg \forall x A )</td>
</tr>
<tr>
<td>1, 2</td>
<td>2</td>
<td>Assume ( \neg \exists x \neg A )</td>
</tr>
<tr>
<td>1, 2, 3</td>
<td>3</td>
<td>Assume ( \neg A )</td>
</tr>
<tr>
<td>1, 2, 3</td>
<td>4</td>
<td>( \exists x \neg A ) ( \exists / 3, x )</td>
</tr>
<tr>
<td>1, 2, 3</td>
<td>5</td>
<td>( \bot ) ( \Rightarrow E 2, 4 )</td>
</tr>
<tr>
<td>1, 2</td>
<td>6</td>
<td>Therefore ( \neg \neg A ) ( \Rightarrow I 3, 5 )</td>
</tr>
<tr>
<td>1, 2</td>
<td>7</td>
<td>( A ) ( Raa 6 )</td>
</tr>
<tr>
<td>1, 2</td>
<td>8</td>
<td>( \forall x A ) ( \forall I 7 )</td>
</tr>
<tr>
<td>1, 2</td>
<td>9</td>
<td>( \bot ) ( \Rightarrow E 1, 8 )</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>Therefore ( \neg \neg \exists x \neg A ) ( \Rightarrow I 2, 9 )</td>
</tr>
</tbody>
</table>

- On line 4: we use \( \neg A = \neg A < x := x > \) and a variable \( x \) is always free for itself in \( A \).
Example 6.1.6 $\neg \forall x A \Rightarrow \exists x \neg A$ (De Morgan’s law)

\[
\begin{array}{ll}
1 & 1 \text{ Assume } \neg \forall x A \\
1, 2 & 2 \text{ Assume } \neg \exists x \neg A \\
1, 2, 3 & 3 \text{ Assume } \neg A \\
1, 2, 3 & 4 \exists x \neg A \quad \exists I \ 3, x \\
1, 2, 3 & 5 \bot \quad \Rightarrow E \ 2, 4 \\
1, 2 & 6 \text{ Therefore } \neg \neg A \quad \Rightarrow I \ 3, 5 \\
1, 2 & 7 A \quad \text{Raa } 6 \\
1, 2 & 8 \forall x A \quad \forall I \ 7 \\
1, 2 & 9 \bot \quad \Rightarrow E \ 1, 8 \\
1 & 10 \text{ Therefore } \neg \neg \exists x \neg A \quad \Rightarrow I \ 2, 9 \\
1 & 11 \exists x \neg A \quad \text{Raa } 10 \\
\end{array}
\]

► On line 4: we use $\neg A = \neg A < x := x >$
and a variable $x$ is always free for itself in $A$. 
Example 6.1.6 $\neg \forall x A \Rightarrow \exists x \neg A$ (De Morgan’s law)

1 1 Assume $\neg \forall x A$
1, 2 2 Assume $\neg \exists x \neg A$
1, 2, 3 3 Assume $\neg A$
1, 2, 3 4 $\exists x \neg A$  $\exists I$ 3, $x$
1, 2, 3 5 $\bot$  $\Rightarrow$  $E$ 2, 4
1, 2 6 Therefore $\neg \neg A$  $\Rightarrow$  $I$ 3, 5
1, 2 7 $A$  Raa 6
1, 2 8 $\forall x A$  $\forall I$ 7
1, 2 9 $\bot$  $\Rightarrow$  $E$ 1, 8
1 10 Therefore $\neg \neg \exists x \neg A$  $\Rightarrow$  $I$ 2, 9
1 11 $\exists x \neg A$  Raa 10
12 Therefore $\neg \forall x A \Rightarrow \exists x \neg A$  $\Rightarrow$  $I$ 1, 11

▶ On line 4: we use $\neg A = \neg A < x : := x >$
and a variable $x$ is always free for itself in $A$. 
### Quantifier rules recap: Figure 6.1

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
</table>
| \[ \frac{A}{\forall x A} \] | \( \forall I \) \( x \) must be free  
  - neither in the environment of the proof,  
  - nor in the context of the premise |
| \[ \frac{\forall x A}{A[x:=t]} \] | \( \forall E \) \( t \) must be free for \( x \) in \( A \) |
| \[ \frac{A[x:=t]}{\exists x A} \] | \( \exists I \) \( t \) must be free for \( x \) in \( A \) |
| \[ \frac{\exists x A}{(A \Rightarrow B)} \] | \( \exists E \) \( x \) must be free  
  - neither in the environment  
  - nor in \( B \),  
  - nor in the context of \( A \Rightarrow B \) |
Overview

Introduction

Rules and examples

Copy rule

Rules for equality

Conclusion
Definition

The copy rule consists in deducing, from a given formula, another formula which is equal up to renaming bound variables.

\[
\frac{A'}{A} \quad \text{copy}
\]
Reminders: Renaming of bound variables (1/3)

Two formulae are $\alpha$-equivalent if one can be transformed into the other by replacing subformulae such as $Qx \ A$ with $Qy \ A < x := y >$ where $Q$ is a quantifier and $y$ does not appear in $Qx \ A$. 

Example 4.4.4

$\forall x \ p(x, z) = \forall y \ p(y, z)$. 

$\forall x \ p(x, z) \neq \forall z \ p(z, z)$. 
Reminders: Renaming of bound variables (1/3)

Two formulae are $\alpha$-equivalent if one can be transformed into the other by replacing subformulae such as $Qx \ A$ with $Qy \ A < x := y >$ where $Q$ is a quantifier and $y$ does not appear in $Qx \ A$.

Example 4.4.4

- $\forall x \ p(x, z) =_\alpha \forall y \ p(y, z)$.
- $\forall x \ p(x, z) \not= _\alpha \forall z \ p(z, z)$. 
Renaming of bound variables (2/3)

Definition 4.4.5
Two formulae are equal up to renaming of bound variables if we can obtain one starting from the other by replacements such as 1

\[ Qx \, A \equiv Qy \, A < x := y > \] where \( y \) is a variable not appearing in \( Qx \, A \)

The two formulae are said to be:

- \( \alpha \)-equivalent
- or a copy of each other
- denoted \( A \equiv_{\alpha} B \)
Renaming of bound variables (3/3)

**Theorem 4.4.6**

If two formulae are equal up to renaming of bound variables then they are equivalent.

**Example 4.4.7**

Let us show that $\forall x \exists y P(x, y)$ and $\forall y \exists x P(y, x)$ are equivalent.
Renaming of bound variables (3/3)

Theorem 4.4.6

If two formulae are equal up to renaming of bound variables then they are equivalent.

Example 4.4.7

Let us show that $\forall x \exists y P(x, y)$ and $\forall y \exists x P(y, x)$ are equivalent.

$\forall x \exists y P(x, y)$
Renaming of bound variables (3/3)

Theorem 4.4.6
If two formulae are equal up to renaming of bound variables then they are equivalent.

Example 4.4.7
Let us show that $\forall x \exists y P(x, y)$ and $\forall y \exists x P(y, x)$ are equivalent.

$\forall x \exists y P(x, y)$

$=_{\alpha} \forall u \exists y P(u, y)$
Renaming of bound variables (3/3)

Theorem 4.4.6

If two formulae are equal up to renaming of bound variables then they are equivalent.

Example 4.4.7

Let us show that $\forall x \exists y P(x, y)$ and $\forall y \exists x P(y, x)$ are equivalent.

\[
\forall x \exists y P(x, y) =_\alpha \forall u \exists y P(u, y) =_\alpha \forall u \exists x P(u, x)
\]
Renaming of bound variables (3/3)

Theorem 4.4.6

If two formulae are equal up to renaming of bound variables then they are equivalent.

Example 4.4.7

Let us show that $\forall x \exists y P(x, y)$ and $\forall y \exists x P(y, x)$ are equivalent.

\[
\begin{align*}
\forall x \exists y P(x, y) & \quad =_{\alpha} \quad \forall u \exists y P(u, y) \\
& \quad =_{\alpha} \quad \forall u \exists x P(u, x) \\
& \quad =_{\alpha} \quad \forall y \exists x P(y, x)
\end{align*}
\]
α-equivalence howto

Technique

- Draw lines between each quantifier and the variables that it binds.
- Erase the name of bound variables.

If after this transformation, the two formulae become identical, then they are α-equivalent.

Example 4.4.8

With the two formulae \( \forall x \exists y P(y, x) \) and \( \forall y \exists x P(x, y) \):

\[
\forall x \exists y P(y, x)
\]
**α-equivalence howto**

**Technique**

- Draw lines between each quantifier and the variables that it binds.
- Erase the name of bound variables.

If after this transformation, the two formulae become identical, then they are α-equivalent.

**Example 4.4.8**

With the two formulae $\forall x \exists y P(y, x)$ and $\forall y \exists x P(x, y)$:

$$\forall x \exists y P(y, x)$$
\(\alpha\)-equivalence howto

**Technique**

- Draw lines between each quantifier and the variables that it binds.
- Erase the name of bound variables.

If after this transformation, the two formulae become identical, then they are \(\alpha\)-equivalent.

**Example 4.4.8**

With the two formulae \(\forall x \exists y P(y, x)\) and \(\forall y \exists x P(x, y)\):

\[
\forall \exists \begin{array}{c}
P(\quad, \quad)
\end{array}
\]
## Exercise

Compute the transformation for

- $A = \forall x \forall y \ R(x, y, y)$
- $B = \forall x \forall y \ R(x, x, y)$

Are $A$ and $B$ $\alpha$-equivalent?
Proof without the copy rule

In the environment \((i) \exists x P(x)\) :
Proof without the copy rule

In the environment $(i) \exists x P(x) :

1 1 Assume $P(x)$
Proof without the copy rule

In the environment \((i) \exists x P(x)\):

1 1 Assume \(P(x)\)
1 2 \(\exists y P(y)\) \(\exists I\ 1, x\)
Proof without the copy rule

In the environment \((i) \exists x P(x)\):

1. 1 Assume \(P(x)\)
2. 2 \(\exists y P(y)\) \(\exists I 1, x\)
3. Therefore \(P(x) \Rightarrow \exists y P(y) \Rightarrow I 1, 2\)
Proof without the copy rule

In the environment \((i) \exists x P(x)\):

1. Assume \(P(x)\)
2. \(\exists y P(y)\) \(\exists I \ 1, \ x\)
3. Therefore \(P(x) \Rightarrow \exists y P(y)\) \(\Rightarrow I \ 1, \ 2\)
4. \(\exists y P(y)\) \(\exists E \ i, \ 3\)
Proof without the copy rule

In the environment $(i) \exists x P(x)$:

1  1  Assume $P(x)$
1  2  $\exists y P(y)$  $\exists I\ 1,\ x$
3  Therefore $P(x) \Rightarrow \exists y P(y)$  $\Rightarrow I\ 1,\ 2$
4  $\exists y P(y)$  $\exists E\ i,\ 3$

Theorem (assumed)

Let $A$ and $A'$ be two formulae which are copies of one another. Then there exists a proof of $A$ in the environment $A'$.

The copy rule is a derivable rule: its use can always be replaced by a (possibly long) proof.

It is the only derivable rule we will allow.
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Reflexivity and congruence

Equality is characterized by two rules:

▶ every term is equal to itself
▶ if two terms are equal, then one can be replaced with the other.
Reflexivity and congruence

Equality is characterized by two rules:

- every term is equal to itself
- if two terms are equal, then one can be replaced with the other.

<table>
<thead>
<tr>
<th>$t = t$</th>
<th>reflexivity</th>
<th>$t$ is a term</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s = t \quad A \prec x := s \succ$</td>
<td>congruence</td>
<td>$s$ and $t$ are two terms free for the variable $x$ in the formula $A$</td>
</tr>
</tbody>
</table>

$s$ and $t$ are two terms free for the variable $x$ in the formula $A$. 

$t = t$ reflexivity $t$ is a term

$s = t \quad A \prec x := s \succ$ congruence $s$ and $t$ are two terms free for the variable $x$ in the formula $A$.
Example 6.1.7

Let us prove that $s = t \Rightarrow t = s$ (symmetry)
Example 6.1.7

Let us prove that $s = t \Rightarrow t = s$ (symmetry)

1 1 Assume $s = t$
Example 6.1.7

Let us prove that $s = t \Rightarrow t = s$ (symmetry)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>Assume $s = t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$s = s$ reflexivity</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>
Example 6.1.7

Let us prove that $s = t \Rightarrow t = s$ (symmetry)

1  1  Assume $s = t$
1  2  $s = s$               reflexivity
1  3  $t = s$               congruence 1, 2
         $s = t$ (x = s) < x := s >
         (x = s) < x := t >
Example 6.1.7

Let us prove that $s = t \Rightarrow t = s$ (symmetry)

<table>
<thead>
<tr>
<th>1</th>
<th>1</th>
<th>Assume $s = t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>$s = s$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>reflexivity</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>$t = s$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>congruence 1, 2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$s = t \ (x = s) &lt; x := s &gt;$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(x = s) &lt; x := t &gt;$</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>Therefore $s = t \Rightarrow t = s \Rightarrow l 1, 3$</td>
</tr>
</tbody>
</table>
Example 6.1.7

Let us prove that $s = t \Rightarrow t = s$ (symmetry)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>Assumption</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>Assume $s = t$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$s = s$</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>$t = s$</td>
</tr>
</tbody>
</table>

Therefore $s = t \Rightarrow t = s \Rightarrow I 1, 3$

Remark: The variable $x$ does not appear in the proof, its only use is to name the place where we replace $s$ with $t$.

In the next examples, we will just underline this place.
Example 6.1.8

Let us prove that $s = t \land t = u \Rightarrow s = u$ (transitivity)
Example 6.1.8

Let us prove that $s = t \land t = u \Rightarrow s = u$ (transitivity)

1 1 Assume $s = t \land t = u$
Example 6.1.8

Let us prove that $s = t \land t = u \Rightarrow s = u$ (transitivity)

1  1  Assume $s = t \land t = u$
1  2  $s = t$  $\land E1$  1

\[ s = u \]  congruence 3, 2

Therefore $s = t \land t = u \Rightarrow s = u \Rightarrow I$  1, 4
Example 6.1.8

Let us prove that $s = t \land t = u \Rightarrow s = u$ (transitivity)

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>Assume $s = t \land t = u$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$s = t$ $\land E1$ 1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>$t = u$ $\land E2$ 1</td>
</tr>
</tbody>
</table>
Example 6.1.8

Let us prove that $s = t \land t = u \Rightarrow s = u$ (transitivity)

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<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>Assume $s = t \land t = u$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$s = t$</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>$t = u$</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>$s = u$</td>
</tr>
</tbody>
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$\land E1$ 1

$\land E2$ 1

congruence 3, 2
Example 6.1.8

Let us prove that \( s = t \land t = u \Rightarrow s = u \) (transitivity)

<table>
<thead>
<tr>
<th>1</th>
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<th>Assume ( s = t \land t = u )</th>
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<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>( s = t )</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>( t = u )</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>( s = u )</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>Therefore ( s = t \land t = u \Rightarrow s = u ) ( \Rightarrow I \ 1, 4 )</td>
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</tbody>
</table>
Overview

Introduction

Rules and examples

Copy rule

Rules for equality

Conclusion
Today

- First-order resolution is **complete**, and one way to build a first-order proof is by **lifting** a propositional proof.
- First-order Natural Deduction
  - New rules for **introducing** and **eliminating** the quantifiers.
  - **Copy**, **equality**
Next lecture

- Tactics
- Consistency of the system