## Organization

**12 weeks:**

- Lecture, 1h30 / week
- Seminar $2 \times 1h30 = 3h / week$

**This week**

- Lecture : today ! and Thursday at 15h15 as usual
- Seminar : 1 session on Wednesday this week, 2 starting next week

https://wackb.gricad-pages.univ-grenoble-alpes.fr/inf402/
Final mark

Evaluations

- Assessments 60%:
  4 periodic tests 10%, midterm exam 20% and project 30%
- Exam: 40%

Project groups: 3-4 students per project group.
- Part 1: Modeling of a logic problem (automated in a software)
- Part 2: Transforming instances of these problems in clauses and solving them using an SAT solver

Examples of problems: N queens, Sudoku-like grids...
Planning

<table>
<thead>
<tr>
<th>Important dates</th>
</tr>
</thead>
<tbody>
<tr>
<td>Winter break: February 24th - March 1st</td>
</tr>
<tr>
<td>Project pre-report: March 6th</td>
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<tr>
<td>Midterm exam: March 9th - 13th</td>
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<td>Spring break: April 20th - 26th</td>
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<td>Project report: April 24th</td>
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<td>Project defense: April 27th - May 7th</td>
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<tr>
<td>Final exam: May 11th - 20th</td>
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<td>Second session: June 15th - 26th</td>
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</tbody>
</table>
Course Material

- Lectures handout (in French, with holes)
- Subject of the project (on the website)
Summary

Prerequisites

Introduction to Logic

Propositional Logic

Syntax

Meaning of formulae (a.k.a. Semantics)

Conclusion
Summary

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Logic

Definitions

- Logic is used to specify what a correct reasoning is, regardless of the application domain.
Logic

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- **Logic** is used to specify what a correct reasoning is, regardless of the application domain.
- A **reasoning** is a way to obtain a conclusion starting from given hypotheses.
Logic

Definitions

- **Logic** is used to specify what a correct reasoning is, regardless of the application domain.
- A **reasoning** is a way to obtain a conclusion starting from given hypotheses.
- A **correct** reasoning does not say anything about the truth of the hypotheses, it only says that **starting from the truth of the hypotheses, one can deduct the truth of the conclusion.**
Examples

Example I

- **Hypothesis I:** All men are mortal
- **Hypothesis II:** Socrates is a man
- **Conclusion:** Socrates is mortal
Examples

Example I

- **Hypothesis I:** All men are mortal
- **Hypothesis II:** Socrates is a man
- **Conclusion:** Socrates is mortal

Example II

- **Hypothesis I:** All that is rare is expensive
- **Hypothesis II:** A cheap horse is rare
- **Conclusion:** A cheap horse is expensive!
Adding a hypothesis
Adding a hypothesis

Example III

- **Hypothesis I**: All that is rare is expensive
- **Hypothesis II**: A cheap horse is rare
- **Hypothesis III**: Every cheap thing is “not expensive”
Adding a hypothesis

Example III

- **Hypothesis I**: All that is rare is expensive
- **Hypothesis II**: A cheap horse is rare
- **Hypothesis III**: Every cheap thing is “not expensive”
- **Conclusion**: Contradictory hypotheses! Since:
  - **Hypothesis I + Hypothesis II**: A cheap horse is expensive
  - **Hypothesis III**: A cheap horse is not expensive
Some history...

- **George Boole** (1815-1864)
  - *symbolic logic*: first try at reasoning without natural language

- **Gottlob Frege** (1848-1925)
  - *propositional calculus*: formal rules for reasoning
  - *proof theory*: a proof itself becomes a mathematical object

- **Bertrand Russell** (1872-1970)
  - *logicism*: attempt at a formalization of all existing mathematics
  - *paradox* found in earlier systems

- **Kurt Gödel** (1906-1978)
  - *completeness* of the first-order predicate calculus
  - *incompleteness theorem* for systems including arithmetic

- **Alonzo Church** (1903-1995)
  - *lambda-calculus*: a proof is an algorithm and vice-versa
Applications

- **Hardware**: logic gates
- **Software verification and correctness**: 
  - Tools: provers COQ, PVS, Prover9, MACE, . . .
  - Meteor (ligne 14)
- **Artificial Intelligence**: 
  - expert system (*MyCin*), ontology
- **Programming**: Prolog
  - artificial intelligence
  - natural language processing
- **Mathematical proofs, Security, . . .**
Course Objectives

- Modeling and formalizing a problem.
- Understanding a formal reasoning, in particular, being able to determine if it is correct or not.
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- Modeling and formalizing a problem.
- Understanding a formal reasoning, in particular, being able to determine if it is correct or not.
- Reasoning, that is, building a correct reasoning using the tools of propositional logic and first order logic.
- Writing a rigorous proof, in particular an induction.
Overview of the Semester

TODAY

- Propositional logic
- Propositional resolution
- Natural deduction for propositional logic

MIDTERM EXAM

- First order logic
- Logical basis for automated proving ("first-order resolution")
- First-order natural deduction

EXAM
Summary

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Propositional Logic

Definition

**Propositional logic** is the logic *without quantifiers*. The only logical operations used are:

- ¬ (negation),
- ∧ (conjunction, also known as logical “and”),
- ∨ (disjunction, also known as logical “or”),
- ⇒ (implication)
- ⇔ (equivalence)
Example: **Formal reasoning**

**Hypotheses:**
- (H1): If Peter is old, then John is not the son of Peter
- (H2): If Peter is not old, then John is the son of Peter
- (H3): If John is Peter’s son then Mary is the sister of John

**Conclusion (C):** Mary is the sister of John, or Peter is old.
Example: **Formal reasoning**

**Hypotheses:**

- **(H1):** If Peter is old, then John is not the son of Peter
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- **(H3):** If John is Peter’s son then Mary is the sister of John

**Conclusion (C):** Mary is the sister of John, or Peter is old.

- \( p \): ”Peter is old”
- \( j \): ”John is the son of Peter”
- \( m \): ”Mary is the sister of John”
Example: **Formal reasoning**

**Hypotheses** :
- (H1): If Peter is old, then John is not the son of Peter
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**Conclusion (C)**: Mary is the sister of John, or Peter is old.

- $p$: ”Peter is old”
- $j$: ”John is the son of Peter”
- $m$: ”Mary is the sister of John”

- (H1): $p \implies \neg j$
- (H2): $\neg p \implies j$
- (H3): $j \implies m$
- (C): $m \lor p$

We prove that $H_1 \land H_2 \land H_3 \implies C$: $(p \implies \neg j) \land (\neg p \implies j) \land (j \implies m) \implies m \lor p$ is true regardless of the truth value of propositions $p$, $j$, $m$. 
Example: **Formal reasoning**

**Hypotheses:**
- (H1): If Peter is old, then John is not the son of Peter
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**Conclusion (C):** Mary is the sister of John, or Peter is old.

- \( p \): ”Peter is old”
- \( j \): ”John is the son of Peter”
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We prove that \( H1 \land H2 \land H3 \Rightarrow C \):

\[
(p \Rightarrow \neg j) \land (\neg p \Rightarrow j) \land (j \Rightarrow m) \Rightarrow m \lor p
\]

is true regardless of the truth value of propositions \( p, j, m \).
Summary

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Conclusion
Vocabulary of the language

- The constants: $\top$ (*true*) and $\bot$ (*false*)
- The variables: for example, $x, y_1$
- The parentheses
- The connectives: $\neg, \lor, \land, \Rightarrow, \Leftrightarrow$
(Strict) Formula

Definition 1.1.1

A strict formula is defined inductively as:

- $\top$ and $\bot$ are strict formulae.
- A variable is a strict formula.
- If $A$ is a strict formula then $\neg A$ is a strict formula.
- If $A$ and $B$ are strict formulae and if $\circ$ is one of the following operations $\lor, \land, \Rightarrow, \Leftrightarrow$ then $(A \circ B)$ is a strict formula.
(Strict) Formula

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- If $A$ and $B$ are strict formulae and if $\circ$ is one of the following operations $\lor, \land, \rightarrow, \leftrightarrow$ then $(A \circ B)$ is a strict formula.

Example 1.1.2

$(a \lor (\neg b \land c))$ is a strict formula, but not $a \lor (\neg b \land c)$, nor $(a \lor (\neg (b) \land c))$. 
Example 1.1.3

The structure of the formula \((a \lor (\neg b \land c))\) is illustrated by the following tree:
Example 1.1.3

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The structure of the formula $\left( a \lor (\neg b \land c) \right)$ is illustrated by the following tree:

```
  ∨
 / \
\a   ∧
 \   \
 □   \neg
```

The diagram clearly shows the logical structure of the formula with $a$, $\neg b$, and $c$ as subformulas, and the disjunction operator $\lor$ connecting them.
Example 1.1.3

The structure of the formula \((a \lor (\neg b \land c))\) is illustrated by the following tree:
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The structure of the formula \((a \lor (\neg b \land c))\) is illustrated by the following tree:

```
  ∨
 ↙   ↙
  a   ∧
        ↘
        ↙
     ¬
    ↓
   b
```

```c
(\neg b \land c)
```

```c
a
```

```c
(\lor)
```

```c
∧
```

```c
(\land)
```

```c
b
```

```c
(\neg)
```

```c
c
```
Exercise

\[((\neg (p \lor q)) \land \neg r)\]
Exercise

\[ ((p \land \neg(p \lor q)) \land \neg r) \]
Size of a formula

Definition 1.1.10

The size of a formula $A$, denoted $|A|$, is inductively defined as:

- $|\top| = 0$ and $|\bot| = 0$.
- If $A$ is a variable then $|A| = 0$.
- $|\neg A| = 1 + |A|$.
- $|(A \circ B)| = |A| + |B| + 1$. 

Example 1.1.11

$|(a \lor (\neg b \land c))| = 3$. 

B. Wack (UGA)
Size of a formula

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Example 1.1.11

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First result

Strict formulae decompose uniquely in their sub-formulae.

Theorem 1.1.13

For every formula $A$, there is one and only one of the following cases:

▶ $A$ is a variable,
▶ $A$ is a constant,
▶ $A$ can be written in a unique manner as $\neg B$ where $B$ is a formula,
▶ $A$ can be written in a unique manner as $(B \circ C)$ where $B$ and $C$ are formulae.

This will allow us to:

▶ prove properties by cases
▶ perform structural induction on the formulae rather than induction on their size.
Prioritized formula

Definition 1.1.14

A prioritized formula is inductively defined in a similar way but:

▶ if $A$ and $B$ are prioritized formulae the $A \circ B$ is a prioritized formula,

▶ if $A$ is a prioritized formula then $(A)$ is a prioritized formula.

Example 1.1.15

$a \lor \neg b \land c$ is a prioritized formula, but not a (strict) formula.
Connective precedence

Definition 1.1.16
By decreasing precedence, the connectives are: ¬, ∧, ∨, ⇒ and ⇔.

Left associativity

For identical connectives, the left-hand side connective has higher precedence:

\[ A \circ B \circ C = (A \circ B) \circ C \]

except for the implication: \( A \Rightarrow B \Rightarrow C = A \Rightarrow (B \Rightarrow C) \)
Example of prioritized formulae

Example 1.1.17

- $a \land b \land c$ is the abbreviation of

- $a \land b \lor c$ is the abbreviation of

- $a \lor b \land c$ is the abbreviation of
Example of prioritized formulae

Example 1.1.17

- $a \land b \land c$ is the abbreviation of $((a \land b) \land c)$
- $a \land b \lor c$ is the abbreviation of $((a \land b) \lor c)$
- $a \lor b \land c$ is the abbreviation of $(a \lor (b \land c))$
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Definition 1.2.1

A truth assignment is a function from the set of variables of a formula to the set \( \{0, 1\} \).

\([A]_v\) denotes the truth value of the formula \( A \) for the assignment \( v \).
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**Example:** Let \(v\) be an assignment such that \(v(x) = 0\) and \(v(y) = 1\). Applying \(v\) to \(x \lor y\) is written as
Truth assignment of a formula

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\([A]_\nu\) denotes the truth value of the formula \(A\) for the assignment \(\nu\).

**Example:** Let \(\nu\) be an assignment such that \(\nu(x) = 0\) and \(\nu(y) = 1\). Applying \(\nu\) to \(x \lor y\) is written as \([x \lor y]_\nu\)

\([x \lor y]_\nu = \)
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\([x \lor y]_\nu = 0 \lor 1 = 1\)
Conclusion:
Definition 1.2.1

A truth assignment is a function from the set of variables of a formula to the set \{0, 1\}. 

\([A]_v\) denotes the truth value of the formula \(A\) for the assignment \(v\).

Example: Let \(v\) be an assignment such that \(v(x) = 0\) and \(v(y) = 1\). Applying \(v\) to \(x \lor y\) is written as \([x \lor y]_v\) 

\([x \lor y]_v = 0 \lor 1 = 1\)

Conclusion: \(x \lor y\) is true for the truth assignment \(v\)
Definition 1.2.2

Let $A$, $B$ be two formulae, $x$ a variable and $v$ a truth assignment.

- $[x]_v =$
- $[\top]_v = [\bot]_v =$
- $[\neg A]_v =$
- $[(A \lor B)]_v =$
- $[(A \land B)]_v =$
- $[(A \Rightarrow B)]_v =$
- $[(A \Leftrightarrow B)]_v =$
Definition 1.2.2

Let $A$, $B$ be two formulae, $x$ a variable and $v$ a truth assignment.

- $[x]_v = v(x)$
- $[\top]_v = 1$, $[\bot]_v = 0$
- $[\neg A]_v = 1 - [A]_v$
- $[(A \lor B)]_v = \max\{[A]_v, [B]_v\}$
- $[(A \land B)]_v = \min\{[A]_v, [B]_v\}$
- $[(A \Rightarrow B)]_v = \begin{cases} 1 & \text{if } [A]_v = 0 \\ [B]_v & \text{otherwise} \end{cases}$
- $[(A \Leftrightarrow B)]_v = \begin{cases} 1 & \text{if } [A]_v = [B]_v \\ 0 & \text{otherwise} \end{cases}$
Truth value of a formula

Definition 1.2.2

Let $A$, $B$ be two formulae, $x$ a variable and $\nu$ a truth assignment.

- $[x]_\nu = \nu(x)$
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- $[\neg A]_\nu =$
- $[(A \lor B)]_\nu =$
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- $[(A \lor B)]_v =$
- $[(A \land B)]_v =$
- $[(A \implies B)]_v =$
- $[(A \iff B)]_v =$
Truth value of a formula

Definition 1.2.2

Let $A$, $B$ be two formulae, $x$ a variable and $\nu$ a truth assignment.

- $[x]_{\nu} = \nu(x)$
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- $[\neg A]_{\nu} = 1 - [A]_{\nu}$
- $[(A \lor B)]_{\nu} = \max\{[A]_{\nu}, [B]_{\nu}\}$
- $[(A \land B)]_{\nu} = $
- $[(A \Rightarrow B)]_{\nu} = $
- $[(A \Leftrightarrow B)]_{\nu} = $
Truth value of a formula

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- $[(A \Rightarrow B)]_{\nu} =$
- $[(A \iff B)]_{\nu} =$
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Truth value of a formula

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- $[(A \Rightarrow B)]_\nu = \text{if } [A]_\nu = 0 \text{ then } 1 \text{ else } [B]_\nu$
- $[(A \Leftrightarrow B)]_\nu = \text{if } [A]_\nu = [B]_\nu \text{ then } 1 \text{ else } 0$
Truth table

Definition 1.2.3

A truth table of a formula $A$ is a table representing the truth values of $A$ for all the possible values of the variables of $A$.

- a line of the truth table = an assignment
- a column of the truth table = the truth values of a formula.
Basic tables

0 indicates false and 1 indicates true.

The value of the constant $\top$ is 1 and the value of the constant $\bot$ is 0

Table 1.1 (truth table of connectives)

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$\neg x$</th>
<th>$x \lor y$</th>
<th>$x \land y$</th>
<th>$x \Rightarrow y$</th>
<th>$x \Leftrightarrow y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
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<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Example:

Example 1.2.4

Give the truth table of the following formulae.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$x \Rightarrow y$</th>
<th>$\neg x$</th>
<th>$\neg x \lor y$</th>
<th>$(x \Rightarrow y) \Leftrightarrow (\neg x \lor y)$</th>
<th>$x \lor \neg y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td></td>
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<tr>
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<td>0</td>
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<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>
Example 1.2.4

Give the truth table of the following formulae.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>( x \implies y )</th>
<th>( \neg x )</th>
<th>( \neg x \lor y )</th>
<th>( (x \implies y) \iff (\neg x \lor y) )</th>
<th>( x \lor \neg y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>( 0 )</td>
<td>( 1 )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
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<td>1</td>
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<th>$\neg x \lor y$</th>
<th>$(x \Rightarrow y) \iff (\neg x \lor y)$</th>
<th>$x \lor \neg y$</th>
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<td>0</td>
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<td>1</td>
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<td></td>
</tr>
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</tr>
</tbody>
</table>
Example:

Example 1.2.4

Give the truth table of the following formulae.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$x \implies y$</th>
<th>$\neg x$</th>
<th>$\neg x \lor y$</th>
<th>$(x \implies y) \iff (\neg x \lor y)$</th>
<th>$x \lor \neg y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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Example:

Example 1.2.4

Give the truth table of the following formulae.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>x ⇒ y</th>
<th>¬x</th>
<th>¬x ∨ y</th>
<th>(x ⇒ y) ⇔ (¬x ∨ y)</th>
<th>x ∨ ¬y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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Equivalent formulae

**Definition 1.2.5**

Two formulae $A$ and $B$ are **equivalent** (denoted $A \equiv B$ or simply $A = B$) if they have the same truth value for **every** assignment.
Equivalent formulae

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Example 1.2.6

$x \Rightarrow y \equiv \neg x \lor y$
Equivalent formulae

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Two formulae $A$ and $B$ are equivalent (denoted $A \equiv B$ or simply $A \equiv B$) if they have the same truth value for every assignment.

Example 1.2.6

$x \Rightarrow y \equiv \neg x \lor y$

Remark:
The logical connective $\Leftrightarrow$ does not mean $A \equiv B$. 
Validity, tautology (1/2)

**Definition 1.2.8**

- A formula is **valid** if its value is 1 for all truth assignments.
- A valid formula is also called a **tautology**.
- Denoted by $|= A$. 

Example 1.2.9

- $(x \Rightarrow y) \iff (\neg x \lor y)$ is valid;
- $x \Rightarrow y$ is not valid since it is false for $x = 1$ and $y = 0$. 

B. Wack (UGA)
Validity, tautology (1/2)

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- $(x \Rightarrow y) \iff (\neg x \lor y)$ is valid;
- $x \Rightarrow y$ is not valid since it is false for $x = 1$ and $y = 0$. 
Valid, tautology (2/2)

Property 1.2.10

The formulae $A$ and $B$ are equivalent ($A \equiv B$) if and only if

\[ A \iff B \text{ is valid.} \]

Proof.

The property is a consequence of the truth table of $\iff$. □
Model for a formula

Definition 1.2.11

A truth assignment $\nu$ for which a formula has truth value equal to 1 is a model for that formula.

$\nu$ satisfies $A$ or $\nu$ makes $A$ true.

Example 1.2.12

A model for $x \Rightarrow y$ is:
Model for a formula

Definition 1.2.11

A truth assignment \( v \) for which a formula has truth value equal to 1 is a \textit{model} for that formula.

\( v \) satisfies \( A \) or \( v \) makes \( A \) true.

Example 1.2.12

A model for \( x \Rightarrow y \) is:

\[
\begin{align*}
x &= 1, \\
y &= 1 \text{ (among others)}
\end{align*}
\]
Model for a formula

**Definition 1.2.11**

A truth assignment \( v \) for which a formula has truth value equal to 1 is a **model** for that formula.

\( v \) **satisfies** \( A \) or \( v \) makes \( A \) **true**.

**Example 1.2.12**

A model for \( x \Rightarrow y \) is:

- \( x = 1, y = 1 \) (among others)

On the opposite, \( x = 1, y = 0 \) is not a model for \( x \Rightarrow y \).
Model for a set of formulae

**Definition 1.2.13**

\( \nu \) is **a model for a set of formulae** \( \{A_1, \ldots, A_n\} \)

if and only if

it is a model for every formula in the set.
Model for a set of formulae

Definition 1.2.13

\( \nu \) is a model for a set of formulae \( \{A_1, \ldots, A_n\} \)
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Example 1.2.14

A model of \( \{a \implies b, b \implies c\} \) is:
Model for a set of formulae

**Definition 1.2.13**

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it is a model for every formula in the set.

**Example 1.2.14**

A model of \( \{a \Rightarrow b, b \Rightarrow c\} \) is:

\[
a = 0, \quad b = 0 \text{ (for any } c)\.
\]
Property of a model for a set of formulae

Property 1.2.15

\( \nu \) is a model for \( \{A_1, \ldots, A_n\} \)
if and only if
\( \nu \) is a model for \( A_1 \land \ldots \land A_n \).
Property of a model for a set of formulae

Property 1.2.15

$v$ is a model for $\{A_1, \ldots, A_n\}$
if and only if
$v$ is a model for $A_1 \land \ldots \land A_n$.

Example 1.2.16

The set of formulae $\{a \Rightarrow b, b \Rightarrow c\}$
and the formula $(a \Rightarrow b) \land (b \Rightarrow c)$
have identical models.
Counter-model

**Definition 1.2.17**

A truth assignment \( v \) which yields the value 0 for a formula is a **counter-model** for the formula.

\( v \) does not satisfy the formula or \( v \) makes the formula false.
Counter-model

**Definition 1.2.17**

A truth assignment $\nu$ which yields the value 0 for a formula is a counter-model for the formula.

$n$ does not satisfy the formula or $n$ makes the formula false.

**Example 1.2.18**

A counter-model of $x \Rightarrow y$ is:
Definition 1.2.17

A truth assignment $\nu$ which yields the value 0 for a formula is a counter-model for the formula.

$\nu$ does not satisfy the formula or $\nu$ makes the formula false.

Example 1.2.18

A counter-model of $x \Rightarrow y$ is:

$x = 1, y = 0.$
Satisfiable formula

Definition 1.2.20
A (set of) formula(e) is satisfiable if it admits a model.

Definition 1.2.21
A (set of) formula(e) is unsatisfiable if it is not satisfiable.
Satisfiable formula

Definition 1.2.20
A (set of) formula(e) is satisfiable if it admits a model.

Definition 1.2.21
A (set of) formula(e) is unsatisfiable if it is not satisfiable.

Example 1.2.22
$x \land \neg x$ is unsatisfiable, but $x \Rightarrow y$ is satisfiable.
Satisfiable formula

Definition 1.2.20
A (set of) formula(e) is satisfiable if it admits a model.

Definition 1.2.21
A (set of) formula(e) is unsatisfiable if it is not satisfiable.

Example 1.2.22
\( x \land \lnot x \) is unsatisfiable, but \( x \Rightarrow y \) is satisfiable.

Beware
unsatisfiable = 0 model  satisfiable = at least 1 model
invalid = at least 1 counter-model  valid = 0 counter-model
Summary

Prerequisites

Introduction to Logic

Propositional Logic

Syntax

Meaning of formulae (a.k.a. Semantics)

Conclusion
Today

- Why define and use formal logic?
- Propositional logic:
  - 1 variable = 1 proposition (a statement) which may be true or false
  - 5 connectives to articulate these propositions
- Meaning of formulae:
  - assignment = choice of a truth value for each variable
  - a formula may be true for 0, 1, several or every assignment
Next time

Homework: build the truth table for the “Peter, John and Mary” example.

- Important equivalences
- Substitutions and replacements
- Normal Forms
Oxford’s motto

The more I study, the more I know
The more I know, the more I forget
The more I forget, the less I know