Organization

12 weeks:

- Lecture, 1h30 / week
- Seminar 2 × 1h30 = 3h / week

This week

- Lecture: today! and Thursday at 15h15 as usual
- Seminar: 1 session on Wednesday this week, 2 starting next week

https://wackb.gricad-pages.univ-grenoble-alpes.fr/inf402/
Final mark

Evaluations

- Assessments 60%:
  - 4 periodic tests 10%, midterm exam 20% and project 30%
- Exam: 40%

Project groups: 3-4 students per project group.

- Part 1: Modeling of a logic problem (automated in a software)
- Part 2: Transforming instances of these problems in clauses and solving them using an SAT solver

Examples of problems: N queens, Sudoku, Takuzu...
Planning

Important dates

- **Winter break**: February 19th - 25th
- **Project pre-report**: March 9th
- **Midterm exam**: March 12th - 16th
- **Spring break**: April 16th - 22nd
- **Project report**: April 27th
- **Project defense**: April 30th - May 4th
- **Final exam**: May 14th - 25th
- **Second session**: June 18th - 22nd
Course Material

- Lectures handout (in French, with holes)
- Subject of the project (on the website)
Summary

Prerequisites

Introduction to Logic

Propositional Logic

Syntax

Meaning of formulae (a.k.a. Semantics)

Conclusion
Summary

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Introduction to Logic

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Logic

Definitions

- Logic is used to specify what a correct reasoning is, regardless of the application domain.
Logic

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- Logic is used to specify what a correct reasoning is, regardless of the application domain.
- A reasoning is a way to obtain a conclusion starting from given hypotheses.
Logic

Definitions

- **Logic** is used to specify what a correct reasoning is, regardless of the application domain.
- A **reasoning** is a way to obtain a conclusion starting from given hypotheses.
- A **correct** reasoning does not say anything about the truth of the hypotheses, it only says that **starting from the truth of the hypotheses, one can deduct the truth of the conclusion.**
Examples

Example I

- **Hypothesis I:** All men are mortal
- **Hypothesis II:** Socrates is a man
- **Conclusion:** Socrates is mortal
Examples

Example I

- **Hypothesis I**: All men are mortal
- **Hypothesis II**: Socrates is a man
- **Conclusion**: Socrates is mortal

Example II

- **Hypothesis I**: All that is rare is expensive
- **Hypothesis II**: A cheap horse is rare
- **Conclusion**: A cheap horse is expensive!
Adding a hypothesis

Hypothesis I: All that is rare is expensive
Hypothesis II: A cheap horse is rare
Hypothesis III: Every cheap thing is "not expensive"

Conclusion: Contradictory hypotheses! Since:
Hypothesis I + Hypothesis II: A cheap horse is expensive
Hypothesis III: A cheap horse is not expensive
Adding a hypothesis

Example III

- **Hypothesis I:** All that is rare is expensive
- **Hypothesis II:** A cheap horse is rare
- **Hypothesis III:** Every cheap thing is “not expensive”
Adding a hypothesis

Example III

- **Hypothesis I:** All that is rare is expensive
- **Hypothesis II:** A cheap horse is rare
- **Hypothesis III:** Every cheap thing is “not expensive”
- **Conclusion:** Contradictory hypotheses! Since:
  - **Hypothesis I + Hypothesis II:** A cheap horse is expensive
  - **Hypothesis III:** A cheap horse is not expensive
Some history...

- **George Boole** (1815-1864)
  - *symbolic logic*: first try at reasoning without natural language

- **Gottlob Frege** (1848-1925)
  - *propositional calculus*: formal rules for reasoning
  - *proof theory*: a proof itself becomes a mathematical object

- **Bertrand Russell** (1872-1970)
  - *logicism*: attempt at a formalization of all existing mathematics
  - *paradox* found in earlier systems

- **Kurt Gödel** (1906-1978)
  - *completeness* of the first-order predicate calculus
  - *incompleteness theorem* for systems including arithmetic

- **Alonzo Church** (1903-1995)
  - *lambda-calculus*: a proof is an algorithm and vice-versa
Applications

- **Hardware**: logic gates
- **Software verification and correctness**:  
  - Tools: provers COQ, PVS, Prover9, MACE, . . .  
  - Meteor (ligne 14)
- **Artificial Intelligence**:  
  - expert system (*MyCin*), ontology
- **Programming**: Prolog  
  - artificial intelligence  
  - natural language processing
- **Mathematical proofs, Security, . . .**
Course Objectives

- Modeling and formalizing a problem.
- Understanding a formal reasoning, in particular, being able to determine if it is correct or not.
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- Reasoning, that is, building a correct reasoning using the tools of propositional logic and first order logic.
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- Modeling and formalizing a problem.
- Understanding a formal reasoning, in particular, being able to determine if it is correct or not.
- Reasoning, that is, building a correct reasoning using the tools of propositional logic and first order logic.
- Writing a rigorous proof, in particular an induction.
Overview of the Semester

TODAY

▶ Propositional logic
▶ Propositional resolution
▶ Natural deduction for propositional logic

MIDTERM EXAM

▶ First order logic
▶ Logical basis for automated proving ("first-order resolution")
▶ First-order natural deduction

EXAM
Summary

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Propositional Logic

Definition

Propositional logic is the logic \textit{without quantifiers}. The only logical operations used are:

\begin{itemize}
  \item $\neg$ (negation),
  \item $\wedge$ (conjunction, also known as logical “and”),
  \item $\lor$ (disjunction, also known as logical “or”),
  \item $\Rightarrow$ (implication)
  \item $\Leftrightarrow$ (equivalence)
\end{itemize}
Example: **Formal reasoning**

**Hypotheses:**

- (H1): If Peter is old, then John is not the son of Peter
- (H2): If Peter is not old, then John is the son of Peter
- (H3): If John is Peter’s son then Mary is the sister of John

**Conclusion (C):** Mary is the sister of John, or Peter is old.
Example: **Formal reasoning**

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- (H1): If Peter is old, then John is not the son of Peter
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**Conclusion** (C): Mary is the sister of John, or Peter is old.

- $p$: ”Peter is old”
- $j$: ”John is the son of Peter”
- $m$: ”Mary is the sister of John”
Example: **Formal reasoning**

**Hypotheses:**
- (H1): If Peter is old, then John is not the son of Peter
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**Conclusion** (C): Mary is the sister of John, or Peter is old.

- $p$: ”Peter is old”
- $j$: ”John is the son of Peter”
- $m$: ”Mary is the sister of John”

\[
\begin{align*}
\text{(H1): } & \ p \Rightarrow \neg j \\
\text{(H2): } & \ 
eg p \Rightarrow j \\
\text{(H3): } & \ j \Rightarrow m \\
\text{(C) : } & \ m \lor p 
\end{align*}
\]
Example: Formal reasoning

Hypotheses:
- (H1): If Peter is old, then John is not the son of Peter
- (H2): If Peter is not old, then John is the son of Peter
- (H3): If John is Peter’s son then Mary is the sister of John

Conclusion (C): Mary is the sister of John, or Peter is old.

- \( p \): "Peter is old"
- \( j \): "John is the son of Peter"
- \( m \): "Mary is the sister of John"

We prove that \( H1 \land H2 \land H3 \Rightarrow C: \)

\[(p \Rightarrow \neg j) \land (\neg p \Rightarrow j) \land (j \Rightarrow m) \Rightarrow m \lor p\]

is true regardless of the truth value of propositions \( p, j, m \).
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Vocabulary of the language

- The constants: $\top$ (true) and $\bot$ (false)
- The variables: for example, $x, y_1$
- The parentheses
- The connectives: $\neg, \lor, \land, \Rightarrow, \Leftrightarrow$
(Strict) Formula

Definition 1.1.1

A strict formula is defined inductively as:

- \(\top\) and \(\bot\) are strict formulae.
- A variable is a strict formula.
- If \(A\) is a strict formula then \(\neg A\) is a strict formula.
- If \(A\) and \(B\) are strict formulae and if \(\circ\) is one of the following operations \(\lor, \land, \Rightarrow, \Leftrightarrow\) then \((A \circ B)\) is a strict formula.
### (Strict) Formula

#### Definition 1.1.1

A strict formula is defined inductively as:

- \( \top \) and \( \bot \) are strict formulae.
- A variable is a strict formula.
- If \( A \) is a strict formula then \( \neg A \) is a strict formula.
- If \( A \) and \( B \) are strict formulae and if \( \circ \) is one of the following operations \( \lor, \land, \Rightarrow, \Leftrightarrow \) then \( (A \circ B) \) is a strict formula.

#### Example 1.1.2

\[(a \lor (\neg b \land c)) \text{ is a strict formula, but not } a \lor (\neg b \land c), \text{ nor } (a \lor (\neg(b) \land c)).\]
Example 1.1.3

The structure of the formula \((a \lor (\neg b \land c))\) is illustrated by the following tree:
Example 1.1.3

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The structure of the formula \((a \lor (\neg b \land c))\) is illustrated by the following tree:
Exercise

\(((p \land \neg(p \lor q)) \land \neg r)\)
Exercise

\[ ((p \land \neg(p \lor q)) \land \neg r) \]
# Size of a formula

## Definition 1.1.10

The **size of a formula** $A$, denoted $|A|$, is inductively defined as:

- $|\top| = 0$ and $|\bot| = 0$.
- If $A$ is a variable then $|A| = 0$.
- $|\neg A| = 1 + |A|$.
- $|(A \circ B)| = |A| + |B| + 1$. 

Example 1.1.11

$|\left( a \lor \left( \neg b \land c \right) \right)| = 3$. 
Size of a formula

Definition 1.1.10

The size of a formula $A$, denoted $|A|$, is inductively defined as:

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Example 1.1.11

$|(a \vee (\neg b \wedge c))| =$
Size of a formula

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Example 1.1.11

$|(a \lor (\neg b \land c))| = 3$. 
First result

Strict formulae decompose uniquely in their sub-formulae.

Theorem 1.1.13

For every formula $A$, there is one and only one of the following cases:

- $A$ is a variable,
- $A$ is a constant,
- $A$ can be written in a unique manner as $\neg B$ where $B$ is a formula,
- $A$ can be written in a unique manner as $(B \circ C)$ where $B$ and $C$ are formulae.

This will allow us to:

- prove properties by cases
- perform structural induction on the formulae rather than induction on their size.
Prioritized formula

Definition 1.1.14

A prioritized formula is inductively defined in a similar way but:

- if $A$ and $B$ are prioritized formulae the $A \circ B$ is a prioritized formula,
- if $A$ is a prioritized formula then $(A)$ is a prioritized formula.

Example 1.1.15

$a \lor \neg b \land c$ is a prioritized formula, but not a (strict) formula.
Connective precedence

**Definition 1.1.16**

By decreasing precedence, the connectives are: \( \neg, \land, \lor, \Rightarrow \) and \( \Leftrightarrow \).

**Left associativity**

For identical connectives, the left-hand side connective has higher precedence:

\[
A \circ B \circ C = (A \circ B) \circ C
\]

except for the implication: \( A \Rightarrow B \Rightarrow C = A \Rightarrow (B \Rightarrow C) \).
Example of prioritized formulae

Example 1.1.17

- $a \land b \land c$ is the abbreviation of $(a \land b) \land c$
- $a \land b \lor c$ is the abbreviation of $(a \land b) \lor c$
- $a \lor b \land c$ is the abbreviation of $a \lor (b \land c)$
Example of prioritized formulae

Example 1.1.17

- $a \land b \land c$ is the abbreviation of $((a \land b) \land c)$
- $a \land b \lor c$ is the abbreviation of $((a \land b) \lor c)$
- $a \lor b \land c$ is the abbreviation of $(a \lor (b \land c))$
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Truth assignment of a formula

Definition 1.2.1

A truth assignment is a function from the set of variables of a formula to the set \{0, 1\}. 

\[ [A]_v \] denotes the truth value of the formula \( A \) for the assignment \( v \).
Truth assignment of a formula

Definition 1.2.1

A truth assignment is a function from the set of variables of a formula to the set \( \{0, 1\} \).

\([A]_\nu\) denotes the truth value of the formula \( A \) for the assignment \( \nu \).

Example: Let \( \nu \) be an assignment such that \( \nu(x) = 0 \) and \( \nu(y) = 1 \).
Applying \( \nu \) to \( x \lor y \) is written as
Truth assignment of a formula

Definition 1.2.1

A truth assignment is a function from the set of variables of a formula to the set $\{0, 1\}$. $[A]_\nu$ denotes the truth value of the formula $A$ for the assignment $\nu$.

Example: Let $\nu$ be an assignment such that $\nu(x) = 0$ and $\nu(y) = 1$. Applying $\nu$ to $x \lor y$ is written as $[x \lor y]_\nu$

$[x \lor y]_\nu =$
Definition 1.2.1

A truth assignment is a function from the set of variables of a formula to the set \{0, 1\}. 

\([A]_\nu\) denotes the truth value of the formula \(A\) for the assignment \(\nu\).

**Example:** Let \(\nu\) be an assignment such that \(\nu(x) = 0\) and \(\nu(y) = 1\). Applying \(\nu\) to \(x \lor y\) is written as \([x \lor y]_\nu\).

\([x \lor y]_\nu = 0 \lor 1 = 1\)

Conclusion:
Truth assignment of a formula

Definition 1.2.1

A truth assignment is a function from the set of variables of a formula to the set \{0, 1\}. \([A]_v\) denotes the truth value of the formula \(A\) for the assignment \(v\).

**Example:** Let \(v\) be an assignment such that \(v(x) = 0\) and \(v(y) = 1\). Applying \(v\) to \(x \lor y\) is written as \([x \lor y]_v\)
\([x \lor y]_v = 0 \lor 1 = 1\)
Conclusion: \(x \lor y\) is true for the truth assignment \(v\)
Truth value of a formula

Definition 1.2.2

Let $A, B$ be two formulae, $x$ a variable and $v$ a truth assignment.

- $[x]_v =$
- $[\top]_v =$, $[\bot]_v =$
- $[\neg A]_v =$
- $[(A \lor B)]_v =$
- $[(A \land B)]_v =$
- $[(A \Rightarrow B)]_v =$
- $[(A \Leftrightarrow B)]_v =$
Truth value of a formula

Definition 1.2.2

Let $A$, $B$ be two formulae, $x$ a variable and $v$ a truth assignment.

- $[x]_v = v(x)$
- $[\top]_v = 1$, $[\bot]_v = 0$
- $[\neg A]_v =$
- $[(A \lor B)]_v =$
- $[(A \land B)]_v =$
- $[(A \Rightarrow B)]_v =$
- $[(A \Leftrightarrow B)]_v =$
Truth value of a formula

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Truth value of a formula

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Let $A$, $B$ be two formulae, $x$ a variable and $v$ a truth assignment.

- $[x]_v = v(x)$
- $[\top]_v = 1$, $[\bot]_v = 0$
- $[\neg A]_v = \neg [A]_v$
- $[(A \lor B)]_v = \max\{[A]_v, [B]_v\}$
- $[(A \land B)]_v = \min\{[A]_v, [B]_v\}$
- $[(A \Rightarrow B)]_v = \begin{cases} 1 & \text{if } [A]_v = 0 \\ [B]_v & \text{else} \end{cases}$
- $[(A \Leftrightarrow B)]_v = \begin{cases} 1 & \text{if } [A]_v = [B]_v \\ 0 & \text{else} \end{cases}$
Definition 1.2.2

Let $A$, $B$ be two formulae, $x$ a variable and $v$ a truth assignment.

- $[x]_v = v(x)$
- $[\top]_v = 1$, $[\bot]_v = 0$
- $[\neg A]_v = 1 - [A]_v$
- $[(A \lor B)]_v = \max\{[A]_v, [B]_v\}$
- $[(A \land B)]_v = \min\{[A]_v, [B]_v\}$
- $[(A \Rightarrow B)]_v = \begin{cases} 1 & \text{if } [A]_v = 0 \\ [B]_v & \text{otherwise} \end{cases}$
- $[(A \iff B)]_v = \begin{cases} 1 & \text{if } [A]_v = [B]_v \\ 0 & \text{otherwise} \end{cases}$
Truth value of a formula

Definition 1.2.2

Let $A, B$ be two formulae, $x$ a variable and $v$ a truth assignment.

- $[x]_v = v(x)$
- $[\top]_v = 1$, $[\bot]_v = 0$
- $[\neg A]_v = 1 - [A]_v$
- $[(A \lor B)]_v = \max\{[A]_v, [B]_v\}$
- $[(A \land B)]_v = $
- $[(A \Rightarrow B)]_v =$
- $[(A \Leftrightarrow B)]_v =$
Truth value of a formula

Definition 1.2.2

Let $A$, $B$ be two formulae, $x$ a variable and $\nu$ a truth assignment.

- $[x]_{\nu} = \nu(x)$
- $[\top]_{\nu} = 1$, $[\bot]_{\nu} = 0$
- $[\neg A]_{\nu} = 1 - [A]_{\nu}$
- $[(A \lor B)]_{\nu} = \max\{[A]_{\nu}, [B]_{\nu}\}$
- $[(A \land B)]_{\nu} = \min\{[A]_{\nu}, [B]_{\nu}\}$
- $[(A \Rightarrow B)]_{\nu} = \frac{1}{1 + [A]_{\nu}}$
- $[(A \Leftrightarrow B)]_{\nu} = \frac{[A]_{\nu} + [B]_{\nu}}{1 + [A]_{\nu}}$
Truth value of a formula

Definition 1.2.2

Let \( A, B \) be two formulae, \( x \) a variable and \( \nu \) a truth assignment.

- \([x]_\nu = \nu(x)\)
- \([\top]_\nu = 1, \,[\bot]_\nu = 0\)
- \([\neg A]_\nu = 1 - [A]_\nu\)
- \([ (A \lor B) ]_\nu = \max\{ [A]_\nu, [B]_\nu \} \)
- \([ (A \land B) ]_\nu = \min\{ [A]_\nu, [B]_\nu \} \)
- \([ (A \Rightarrow B) ]_\nu = \text{if } [A]_\nu = 0 \text{ then } 1 \text{ else } [B]_\nu \)
- \([ (A \Leftrightarrow B) ]_\nu = \ldots\)
Truth value of a formula

Definition 1.2.2

Let $A$, $B$ be two formulae, $x$ a variable and $\nu$ a truth assignment.

- $[x]_{\nu} = \nu(x)$
- $[\top]_{\nu} = 1$, $[\bot]_{\nu} = 0$
- $[\neg A]_{\nu} = 1 - [A]_{\nu}$
- $[(A \lor B)]_{\nu} = \max\{[A]_{\nu}, [B]_{\nu}\}$
- $[(A \land B)]_{\nu} = \min\{[A]_{\nu}, [B]_{\nu}\}$
- $[(A \Rightarrow B)]_{\nu} = $ if $[A]_{\nu} = 0$ then 1 else $[B]_{\nu}$
- $[(A \Leftrightarrow B)]_{\nu} = $ if $[A]_{\nu} = [B]_{\nu}$ then 1 else 0
Truth table

Definition 1.2.3

A truth table of a formula $A$ is a table representing the truth values of $A$ for all the possible values of the variables of $A$.

- a line of the truth table = an assignment
- a column of the truth table = the truth values of a formula.
Basic tables

0 indicates false and 1 indicates true.
The value of the constant $\top$ is 1 and the value of the constant $\bot$ is 0

Table 1.1 (truth table of connectives)

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$\neg x$</th>
<th>$x \lor y$</th>
<th>$x \land y$</th>
<th>$x \Rightarrow y$</th>
<th>$x \Leftrightarrow y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
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<tr>
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<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Example:

Example 1.2.4

Give the truth table of the following formulae.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$x \Rightarrow y$</th>
<th>$\neg x$</th>
<th>$\neg x \lor y$</th>
<th>$(x \Rightarrow y) \Leftrightarrow (\neg x \lor y)$</th>
<th>$x \lor \neg y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
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<td>1</td>
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<td>1</td>
<td>1</td>
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</tr>
</tbody>
</table>
Example:

Example 1.2.4

Give the truth table of the following formulae.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>x ⇒ y</th>
<th>¬x</th>
<th>¬x ∨ y</th>
<th>(x ⇒ y) ⇔ (¬x ∨ y)</th>
<th>x ∨ ¬y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>¬x</td>
<td>¬x ∨ y</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>¬x</td>
<td>¬x ∨ y</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>¬x</td>
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</table>
Example:

**Example 1.2.4**

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<th>y</th>
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<th>¬x</th>
<th>¬x \lor y</th>
<th>(x ⇒ y) ⇔ (¬x \lor y)</th>
<th>x \lor ¬y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
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</tr>
</tbody>
</table>
Example:

Example 1.2.4

Give the truth table of the following formulae.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$x \Rightarrow y$</th>
<th>$\neg x$</th>
<th>$\neg x \lor y$</th>
<th>$(x \Rightarrow y) \iff (\neg x \lor y)$</th>
<th>$x \lor \neg y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
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<td>1</td>
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</table>
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<th>$\neg x$</th>
<th>$\neg x \lor y$</th>
<th>$(x \Rightarrow y) \Leftrightarrow (\neg x \lor y)$</th>
<th>$x \lor \neg y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
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<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Example 1.2.4

Give the truth table of the following formulae.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>x ⇒ y</th>
<th>¬x</th>
<th>¬x ∨ y</th>
<th>(x ⇒ y) ⇔ (¬x ∨ y)</th>
<th>x ∨ ¬y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<td>1</td>
</tr>
</tbody>
</table>
Equivalent formulae

**Definition 1.2.5**

Two formulae $A$ and $B$ are equivalent (denoted $A \equiv B$ or simply $A = B$) if they have the same truth value for every assignment.
Equivalent formulae

Definition 1.2.5

Two formulae \( A \) and \( B \) are equivalent (denoted \( A \equiv B \) or simply \( A \equiv B \)) if they have the same truth value for every assignment.

Example 1.2.6

\[ x \implies y \equiv \neg x \lor y \]
Equivalent formulae

Definition 1.2.5

Two formulae $A$ and $B$ are equivalent (denoted $A \equiv B$ or simply $A = B$) if they have the same truth value for every assignment.

Example 1.2.6

$x \Rightarrow y \equiv \neg x \lor y$

Remark:
The logical connective $\Leftrightarrow$ does not mean $A \equiv B$. 
Validity, tautology (1/2)

Definition 1.2.8

- A formula is **valid** if its value is 1 for all truth assignments.
- A valid formula is also called a **tautology**.
- Denoted by $\models A$. 

Example 1.2.9

$$(x \implies y) \iff (\neg x \lor y)$$ is valid;

$x \implies y$ is not valid since it is false for $x = 1$ and $y = 0$. 

Definition 1.2.8

- A formula is **valid** if its value is 1 for all truth assignments.
- A valid formula is also called a **tautology**.
- Denoted by \( \models A \).

Example 1.2.9

- \((x \Rightarrow y) \iff (\neg x \lor y)\) is valid;
- \(x \Rightarrow y\) is not valid since it is false for \(x = 1\) and \(y = 0\).
Valid, tautology (2/2)

Property 1.2.10

The formulae $A$ and $B$ are equivalent ($A \equiv B$) if and only if formula $A \iff B$ is valid.

Proof.

The property is a consequence of the truth table of $\iff$. 

□
Model for a formula

**Definition 1.2.11**

A truth assignment \( v \) for which a formula has truth value equal to 1 is a **model** for that formula.

\( v \) **satisfies** \( A \) or \( v \) **makes** \( A \) **true**.

**Example 1.2.12**

A model for \( x \Rightarrow y \) is:
Model for a formula

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A model for \( x \Rightarrow y \) is:

\[
x = 1, \ y = 1 \text{ (among others)}
\]
Model for a formula

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**Example 1.2.12**

A model for \( x \Rightarrow y \) is:

\[
\begin{align*}
&x = 1, \ y = 1 \ (\text{among others}) \\
&
\end{align*}
\]

On the opposite, \( x = 1, \ y = 0 \) is not a model for \( x \Rightarrow y \).
Model for a set of formulae

Definition 1.2.13

$v$ is a model for a set of formulae $\{A_1, \ldots, A_n\}$ if and only if it is a model for every formula in the set.
Model for a set of formulae

Definition 1.2.13

ν is a model for a set of formulae \( \{A_1, \ldots, A_n\} \)
if and only if
it is a model for every formula in the set.

Example 1.2.14

A model of \( \{a \Rightarrow b, b \Rightarrow c\} \) is:
Definition 1.2.13

\( \nu \) is a model for a set of formulae \( \{A_1, \ldots, A_n\} \)
if and only if
it is a model for every formula in the set.

Example 1.2.14

A model of \( \{a \Rightarrow b, b \Rightarrow c\} \) is:

\[ a = 0, b = 0 \text{ (for any } c) . \]
Property of a model for a set of formulae

Property 1.2.15

\( \nu \) is a model for \( \{ A_1, \ldots, A_n \} \) if and only if
\( \nu \) is a model for \( A_1 \land \ldots \land A_n \).
Property of a model for a set of formulae

Property 1.2.15

\( \nu \) is a model for \( \{ A_1, \ldots, A_n \} \)
if and only if
\( \nu \) is a model for \( A_1 \land \ldots \land A_n \).

Example 1.2.16

The set of formulae \( \{ a \Rightarrow b, b \Rightarrow c \} \)
and the formula \( (a \Rightarrow b) \land (b \Rightarrow c) \)
have identical models.
Counter-model

**Definition 1.2.17**

A truth assignment $\nu$ which yields the value 0 for a formula is a **counter-model** for the formula.

$\nu$ does not satisfy the formula or $\nu$ makes the formula false.
Counter-model

Definition 1.2.17

A truth assignment \( \nu \) which yields the value 0 for a formula is a counter-model for the formula.

\( \nu \) does not satisfy the formula or \( \nu \) makes the formula false.

Example 1.2.18

A counter-model of \( x \Rightarrow y \) is:
Counter-model

Definition 1.2.17

A truth assignment $\nu$ which yields the value 0 for a formula is a counter-model for the formula.

$\nu$ does not satisfy the formula or $\nu$ makes the formula false.

Example 1.2.18

A counter-model of $x \Rightarrow y$ is:

$x = 1, y = 0.$
Satisfiable formula

**Definition 1.2.20**

A (set of) formula(e) is **satisfiable** if it admits a model.

**Definition 1.2.21**

A (set of) formula(e) is **unsatisfiable** if it is not satisfiable.
Satisfiable formula

Definition 1.2.20
A (set of) formula(e) is **satisfiable** if it admits a model.

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Example 1.2.22
$x \land \neg x$ is unsatisfiable, but $x \Rightarrow y$ is satisfiable.
Satisfiable formula

Definition 1.2.20
A (set of) formula(e) is **satisfiable** if it admits a model.

Definition 1.2.21
A (set of) formula(e) is **unsatisfiable** if it is not satisfiable.

Example 1.2.22
\[ x \land \neg x \] is unsatisfiable, but \[ x \Rightarrow y \] is satisfiable.

Beware
unsatisfiable = 0 model       satisfiable = at least 1 model
invalid = at least 1 counter-model  valid = 0 counter-model
Summary

Prerequisites

Introduction to Logic

Propositional Logic

Syntax

Meaning of formulae (a.k.a. Semantics)

Conclusion
Today

- Why define and use formal logic?
- Propositional logic:
  - 1 variable = 1 proposition (a statement) which may be true or false
  - 5 connectives to articulate these propositions
- Meaning of formulae:
  - assignment = choice of a truth value for each variable
  - a formula may be true for 0, 1, several or every assignment
Next time

Homework: build the truth table for the “Peter, John and Mary” example.

- Important equivalences
- Substitutions and replacements
- Normal Forms
Oxford’s motto

The more I study, the more I know
The more I know, the more I forget
The more I forget, the less I know