Transformations of logical formulae

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Previous lecture

- Why formal logic ?
- Propositional logic
- Syntax
- Meaning of formulae

Our example with a truth table

Hypotheses:

- (H1): If Peter is old, then John is not the son of Peter
- (H2): If Peter is not old, then John is the son of Peter
- (H3): If John is Peter's son then Mary is the sister of John

Conclusion (C): Mary is the sister of John, or Peter is old.

$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

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p	j	m	$p \Rightarrow \neg j$	$\neg p \Rightarrow j$	$j \Rightarrow m$	$H_1 \wedge H_2 \wedge H_3$	$m \vee p$	$H_1 \wedge H_2 \wedge H_3 \Rightarrow m \vee p$
0	0	0	1	0	1	0	0	1
0	0	1	1	0	1	0	1	1
0	1	0	1	1	0	0	0	1
0	1	1	1	1	1	1	1	1
1	0	0	1	1	1	1	1	1
1	0	1	1	1	1	1	1	1
1	1	0	0	1	0	0	1	1
1	1	1	0	1	1	0	1	1

Plan

Plan

Logical consequence (entailment)

Definition 1.2.24

A is a consequence of the set Γ of hypotheses ($\Gamma \models A$) if every model of Γ is a model of A.

Remark 1.2.26

$\models A$ denotes that A is valid.
(Every truth assignment is a model for the empty set.)

Example of a consequence

Example 1.2.28

$a \Rightarrow b, b \Rightarrow c \models a \Rightarrow c$.

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0	1	0	1	0	1
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	0	1	1
1	1	0	1	0	0
1	1	1	1	1	1

ESSENTIAL property

Often used in exercises and during exams.

Property 1.2.27

Let $H_n = A_1 \wedge \ldots \wedge A_n$.
The following three formulations are equivalent:

1. $A_1, \ldots, A_n \models B$
2. $H_n \Rightarrow B$ is valid.
3. $H_n \wedge \neg B$ is unsatisfiable.

Proof.

Based on the truth tables of the connectives.
We prove that $1 \Rightarrow 2$ then $2 \Rightarrow 3$ and $3 \Rightarrow 1$. $\square$

Proof (1/3)

- $1 \Rightarrow 2$: let us assume that $A_1, \ldots, A_n \models B$.

Let v be a truth assignment:

- if v is not a model for $A_1, \ldots, A_n$:
  for a certain i we have $[A_i]_v = 0$, hence $[H_n]_v = 0$.
  Thus $[H_n \Rightarrow B]_v = 1$.
- is v is a model for $A_1, \ldots, A_n$:
  then by hypothesis v is a model for B therefore $[B]_v = 1$.
  Thus $[H_n \Rightarrow B]_v = 1$.

Therefore $H_n \Rightarrow B$ is valid.

Proof (2/3)

- $2 \Rightarrow 3$: let us assume that $H_n \Rightarrow B$ is valid.
  For every truth assignment v:
  - either $[H_n]_v = 0$,
  - or $[H_n]_v = 1$ and $[B]_v = 1$.

  However $[H_n \wedge \neg B]_v = \min([H_n]_v, [\neg B]_v) = \min([H_n]_v, 1 - [B]_v)$.

  In both cases, we have $[H_n \wedge \neg B]_v = 0$.
  Therefore $H_n \wedge \neg B$ is unsatisfiable.

Proof (3/3)

- $3 \Rightarrow 1$: let us assume that $H_n \wedge \neg B$ is unsatisfiable.
  Let us show that $A_1, \ldots, A_n \models B$.

  Let v be a truth assignment model of $A_1, \ldots, A_n$:
  - $[H_n]_v = [A_1 \wedge \ldots \wedge A_n]_v = 1$.
  - According to our hypothesis $[\neg B]_v = 0$.
    Hence, $1 - [B]_v = 0$ so $[B]_v = 1$, i.e. v is a model for B.

Exercise 7 shows why proving these 3 circular implications is sufficient.

Instance of the property

Example 1.2.28

a	b	c	$a \Rightarrow b$	$b \Rightarrow c$	$a \Rightarrow c$	$(a \Rightarrow b) \wedge (b \Rightarrow c)$ $\Rightarrow (a \Rightarrow c)$	$(a \Rightarrow b) \wedge (b \Rightarrow c)$ $\wedge \neg(a \Rightarrow c)$
0	0	0	1	1	1		
0	0	1	1	1	1		
0	1	0	1	0	1		
0	1	1	1	1	1		
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1	1	0	1	0	0	1	0
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Compactness

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A set of propositional formulae has a model if an only if every finite subset of it has a model.

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Theorem 1.2.30 Propositional compactness

A set of propositional formulae has a model if an only if every finite subset of it has a model.

This theorem may look trivial. However, the set of formulae may be infinite !

This result will be used at a later stage in the course (bases for automated theorem proving).

Plan

Preamble

How to prove that a formula is valid?

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- Truth table
  - Problem: for a formula having 100 variables, the truth table will contain 2^{100} lines (unable to be computed, even by a computer!).
- Idea:
  - Simplify the formula using transformations
  - Then, study the simplified formula using truth tables or a logic reasoning

Disjunction

- associative $x \vee (y \vee z) \equiv (x \vee y) \vee z$
- commutative $x \vee y \equiv y \vee x$
- idempotent $x \vee x \equiv x$

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Ditto for conjunction.

Distributivity

- Conjunction distributes over disjunction
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Neutrality and Absorption

- 0 is the neutral element for disjunction $0 \vee x \equiv x$
- 1 is the neutral element for conjunction $1 \wedge x \equiv x$
- 1 is the absorbing element for disjunction $1 \vee x \equiv 1$
- 0 is the absorbing element for conjunction $0 \wedge x \equiv 0$

Negation

- Negation laws:
  - $x \wedge \neg x \equiv 0$
  - $x \vee \neg x \equiv 1$ (The law of excluded middle)
- $\neg\neg x \equiv x$
- $\neg 0 \equiv 1$
- $\neg 1 \equiv 0$

De Morgan laws

- $\neg(x \wedge y) \equiv \neg x \vee \neg y$
- $\neg(x \vee y) \equiv \neg x \wedge \neg y$

Augustus De Morgan (1860) builds on Boole's algebra:

- Work about quantifiers
- Calculus of relations
  (also see C.S. Peirce's works)

which laid grounds for first-ordre logic (see 2nd part of the course).

- Notion of duality in Boole's algebras

expressed in particular as De Morgan's laws

- Involved (though very briefly) in the first conjectures about the four colour theorem

Simplification laws

Property 1.2.31

For every x, y we have:

- $x \vee (x \wedge y) \equiv x$
- $x \wedge (x \vee y) \equiv x$
- $x \vee (\neg x \wedge y) \equiv x \vee y$

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Substitution

Definition 1.3.1

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Example: $A = \neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$

- Let σ the following substitution: $\sigma(p) = (a \vee b), \sigma(q) = (c \wedge d)$
- $A\sigma =$

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- Let σ the following substitution: $\sigma(p) = (a \vee b), \sigma(q) = (c \wedge d)$
- $A\sigma = \neg((a \vee b) \wedge (c \wedge d)) \Leftrightarrow (\neg(a \vee b) \vee \neg(c \wedge d))$

Finite support substitution

Definition 1.3.2

- The support of a substitution σ is the set of variables x such that $x\sigma \neq x$.
- A finite support substitution σ is denoted $< x_1 := A_1, \ldots, x_n := A_n >$

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Example 1.3.3

$A = x \vee x \wedge y \Rightarrow z \wedge y$ and $\sigma = < x := a \vee b, z := b \wedge c >$

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Example 1.3.3

$A = x \vee x \wedge y \Rightarrow z \wedge y$ and $\sigma =< x := a \vee b, z := b \wedge c >$

$$\boxed{A\sigma = (a \vee b) \vee (a \vee b) \wedge y \Rightarrow (b \wedge c) \wedge y}$$

Properties of substitutions

Property 1.3.4

Let v be a truth assignment and σ a substitution.
Let w be the assignment $w : x \mapsto [\sigma(x)]_v$.
For any formula A, we have $[A\sigma]_v = [A]_w$.

Properties of substitutions

Property 1.3.4

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Example 1.3.5 :

Let $A = x \vee y \vee d$
Let $\sigma =< x := a \vee b, y := b \wedge c >$
Let v be $v(a) = 1$, $v(b) = 0$, $v(c) = 0$, $v(d) = 0$

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$$A\sigma = (a \vee b) \vee (b \wedge c) \vee d$$

$$\begin{aligned}[A\sigma]_v &= (1 \vee 0) \vee (0 \wedge 0) \vee 0 \\ &= 1 \vee 0 \vee 0 = 1\end{aligned}$$

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$$A\sigma = (a \vee b) \vee (b \wedge c) \vee d$$

$$\begin{aligned} w(x) &= [a \vee b]_v = 1 \vee 0 = 1 \\ w(y) &= [b \wedge c]_v = 0 \wedge 0 = 0 \\ w(d) &= [d]_v = 0 \end{aligned}$$

$$\begin{aligned} [A\sigma]_v &= (1 \vee 0) \vee (0 \wedge 0) \vee 0 \\ &= 1 \vee 0 \vee 0 = 1 \end{aligned}$$

$$[A]_w = 1 \vee 0 \vee 0 = 1$$

Initial step: $|A| = 0$

Two possible cases:

- If A is $\top$ or $\perp$ then $A\sigma = A$ and $[A]_v$ does not depend on v.
- If A is a variable x, then by construction $[x\sigma]_v == w(x)$.

Induction

Hypothesis: Assume the property holds for any formula of height less or equal to n.
Let A be a formula of height $n+1$; there are two possible cases:

- Case 1: $A = \neg B$ with $|B| = n$.
  $[A\sigma]_v = [\neg B\sigma]_v = [\neg(B\sigma)]_v = 1 - [B\sigma]_v$ and
  $[A]_w = [\neg B]_w = 1 - [B]_w$.
  Since $|B| = n$, by induction hypothesis $[B\sigma]_v = [B]_w$
  Hence, $[A\sigma]_v = [A]_w$.

Induction

Hypothesis: Assume the property is true for any formula of height less or equal to n.
Let A be a formula of height $n+1$; there are two possible cases:

- Case 2: $A = (B \circ C)$ with $|B| < n+1$ and $|C| < n+1$.
  Then $[A\sigma]_v = [B\sigma \circ C\sigma]_v$
  and $[A]_w = [B \circ C]_w$

  By induction hypothesis $[B\sigma]_v = [B]_w$ and $[C\sigma]_v = [C]_w$.
  Since the semantics for $\circ$ remain the same, $[A\sigma]_v = [A]_w$.

Substitution of a valid formula

Theorem 1.3.6

If A is valid then $A\sigma$ is valid for any σ.

Proof.

$\square$

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Proof.

Let v be any truth assignment.

According to property 1.3.4 : $[A\sigma]_v = [A]_w$ where $w(x) = [\sigma(x)]_v$.

Since A is valid, $[A]_w = 1$.

Consequently, $A\sigma$ equals 1 in every truth assignment, therefore $A\sigma$ is a valid formula. $\square$

Examples

Example 1.3.7

Let A be the formula $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$. This formula is valid, it is an important equivalence. Let σ the following substitution: $< p := (a \vee b), q := (c \wedge d) >$. The formula

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$\boxed{A\sigma = \neg((a \vee b) \wedge (c \wedge d)) \Leftrightarrow (\neg(a \vee b) \vee \neg(c \wedge d)) \text{ is also valid.}}$

Replacement

Definition 1.3.8

The formula D is obtained by replacing certain occurrences of A by B in C if:

- C can be written $E < x := A >$
- D can be written $E < x := B >$

for some formula E.

Examples

Example 1.3.9

Consider the formula $C = ((a \Rightarrow b) \vee \neg(a \Rightarrow b))$.

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Properties of the replacements

Theorem 1.3.10

If D is obtained by replacing, in C, some occurrences of A by B, then $(A \Leftrightarrow B) \models (C \Leftrightarrow D)$.

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By definition, $C = E < x := A >$ et $D = E < x := B >$.
Assume that $[A]_v = [B]_v$, then w is the same for both substitutions.
Therefore $[C]_v = [D]_v$: the assignment v is a model of $(C \Leftrightarrow D)$. $\square$

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Example 1.3.12: $p \Leftrightarrow q \models (p \vee (\boxed{p} \Rightarrow r)) \Leftrightarrow (p \vee (\boxed{q} \Rightarrow r))$.

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Example 1.3.12: $p \Leftrightarrow q \models (p \vee (\boxed{p} \Rightarrow r)) \Leftrightarrow (p \vee (\boxed{q} \Rightarrow r))$.

Corollary 1.3.11

Let D be obtained by replacing, in C, one occurrence of A by B.
If $A \equiv B$ then $C \equiv D$.

Plan

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- The monomial $x \wedge \neg y \wedge z \wedge \neg x$ contains x and $\neg x$: its value is 0.

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- $x, y, \neg z$ are literals.
- $x \wedge \neg y \wedge z$ is a monomial
- The monomial $x \wedge \neg y \wedge z \wedge \neg x$ contains x and $\neg x$: its value is 0.
- $x \vee \neg y \vee z$ is a clause
- The clause $x \vee \neg y \vee z \vee \neg x$ contains x and $\neg x$: its value is 1.

Normal form

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A formula is in normal form if it only contains the operators $\wedge, \vee, \neg$ and the negations are only applied to variables.

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Example 1.4.4

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$a \Rightarrow b$ is not in normal form, even if it is equivalent to the first one.

Every formula admits an equivalent normal form.

Computing a normal form

1. **Equivalence elimination**

2. **Implication elimination**

3. **Shifting negations towards variables**

Computing a normal form

1. **Equivalence elimination**
   Replace any occurrence of $A \Leftrightarrow B$ by
   (a) $(\neg A \vee B) \wedge (\neg B \vee A)$
   OR
   (b) $(A \wedge B) \vee (\neg A \wedge \neg B)$
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   Replace any occurrence of $A \Rightarrow B$ by $\neg A \vee B$
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   (a) $(\neg A \vee B) \wedge (\neg B \vee A)$
   OR
   (b) $(A \wedge B) \vee (\neg A \wedge \neg B)$
2. **Implication elimination**
   Replace any occurrence of $A \Rightarrow B$ by $\neg A \vee B$
3. **Shifting negations towards variables**
   Replace any occurrence of
   (a) $\neg\neg A$ by A
   (b) $\neg(A \vee B)$ by $\neg A \wedge \neg B$
   (c) $\neg(A \wedge B)$ by $\neg A \vee \neg B$

Remark 1.4.5: simplifications

Simplify as soon as possible:

1. Replace $\neg(A \Rightarrow B)$ by $A \wedge \neg B$.

Remark 1.4.5: simplifications

Simplify as soon as possible:

1. Replace $\neg(A \Rightarrow B)$ by $A \wedge \neg B$.
2. Replacing a conjunction by 0 if it contains a formula and its negation
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5. Apply the simplifications:
   - $x \vee (x \wedge y) \equiv x$,
   - $x \wedge (x \vee y) \equiv x$,
   - $x \vee (\neg x \wedge y) \equiv x \vee y$

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A formula is in disjunctive normal form (DNF) if and only if it is a disjunction (sum) of monomials.

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Example 1.4.7

$(x \land y) \lor (\neg x \land \neg y \land z)$ is a DNF, which has two main models:

- $x \mapsto 1, y \mapsto 1$
- $x \mapsto 0, y \mapsto 0, z \mapsto 1$

Conjunctive normal form (CNF)

Definition 1.4.11

A formula is a conjunctive normal form (CNF) if and only if it is a conjunction (product) of clauses.

Apply the (unusual) distributivity of disjunction over conjunction:

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- $x \mapsto 0, y \mapsto 0$
- $x \mapsto 1, y \mapsto 1, z \mapsto 0.$

Examples 1.4.8 and 1.4.13

Transformation in DNF of the following:

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Transformation in DNF of the following:

$$(a \lor b) \land (c \lor d \lor e) \equiv$$

$$\boxed{(a \land c) \lor (a \land d) \lor (a \land e) \lor (b \land c) \lor (b \land d) \lor (b \land e).}$$

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We transform $\neg A$ in an equivalent disjunction of monomials B:

- If $B = 0$ then $\neg A = 0$, hence $A = 1$, that is, A is valid
- Otherwise B is equal to a disjunction of nonzero monomials equivalent to $\neg A$, which give us models of $\neg A$, hence counter-models of A.

Example 1.4.9

Let $A = (p \Rightarrow (q \Rightarrow r)) \Rightarrow (p \wedge q \Rightarrow r)$

Determine whether A is valid.

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$\equiv (p \Rightarrow (q \Rightarrow r)) \wedge \neg(p \wedge q \Rightarrow r)$ since $\neg(B \Rightarrow C) \equiv B \wedge \neg C$
$\equiv (\neg p \vee \neg q \vee r) \wedge \neg(p \wedge q \Rightarrow r)$ eliminating two $\Rightarrow$

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\equiv (p \Rightarrow (q \Rightarrow r)) \wedge \neg(p \wedge q \Rightarrow r) & \text{since } \neg(B \Rightarrow C) \equiv B \wedge \neg C \\
\equiv (\neg p \vee \neg q \vee r) \wedge \neg(p \wedge q \Rightarrow r) & \text{eliminating two } \Rightarrow \\
\equiv (\neg p \vee \neg q \vee r) \wedge (p \wedge q \wedge \neg r) & \text{since } \neg(B \Rightarrow C) \equiv B \wedge \neg C
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\equiv (\neg q \vee r) \wedge p \wedge q \wedge \neg r & \text{simplification } x \wedge (\neg x \vee y)
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\equiv (\neg p \vee \neg q \vee r) \wedge (p \wedge q \wedge \neg r) & \text{since } \neg(B \Rightarrow C) \equiv B \wedge \neg C \\
\equiv (\neg q \vee r) \wedge p \wedge q \wedge \neg r & \text{simplification } x \wedge (\neg x \vee y) \\
\equiv (r) \wedge p \wedge q \wedge \neg r & \text{simplification } x \wedge (\neg x \vee y)
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\equiv (r) \wedge p \wedge q \wedge \neg r & \text{simplification } x \wedge (\neg x \vee y) \\
= 0 & \text{since we have } r \wedge \neg r \text{ in the monomial}
\end{array}$$

Hence $\neg A = 0$ and $A = 1$, that is A is valid.

Example 1.4.10

Let $A = (a \Rightarrow b) \wedge c \vee (a \wedge d)$.

Determine whether A is valid.

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$\neg A$
$\equiv \neg((a \Rightarrow b) \wedge c) \wedge \neg(a \wedge d)$ (de Morgan)

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Determine whether A is valid.

$$\begin{aligned}
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&\equiv (\neg(a \Rightarrow b) \vee \neg c) \wedge (\neg a \vee \neg d) && \text{(de Morgan)}
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 & \text{distributivity of } \vee \text{ over } \wedge
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$\equiv (a \wedge \neg b \wedge \neg a) \vee (a \wedge \neg b \wedge \neg d) \vee (\neg c \wedge \neg a) \vee (\neg c \wedge \neg d)$	
	distributivity of $\vee$ over $\wedge$
$\equiv (a \wedge \neg b \wedge \neg d) \vee (\neg c \wedge \neg a) \vee (\neg c \wedge \neg d)$	1st monomial contradictory

We obtain 3 models of $\neg A$: $(a \mapsto 1, b \mapsto 0, d \mapsto 0)$, $(a \mapsto 0, c \mapsto 0)$, $(c \mapsto 0, d \mapsto 0)$.
That is, counter-models of A.
Hence A is not valid.

Plan

Today

- Substitutions allow us to **deduce the validity** of a formula from another
- Replacements allow us to change part of a formula **without changing its meaning** and thus allow us to compute a simpler equivalent formula
- Every formula admits normal forms which allow to **highlight its models** and counter-models

Next course

- Boolean algebra
- Boolean functions
- Resolution

Prove our example by formula simplification

$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$