Transformations of logical formulae

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Previous lecture

- Why formal logic?
- Propositional logic
- Syntax
- Meaning of formulae
Our example with a truth table

Hypotheses:
- (H1): If Peter is old, then John is not the son of Peter
- (H2): If Peter is not old, then John is the son of Peter
- (H3): If John is Peter’s son then Mary is the sister of John

Conclusion (C): Mary is the sister of John, or Peter is old.

\[(p \implies \neg j) \land (\neg p \implies j) \land (j \implies m) \implies m \lor p\]
Our example with a truth table

Hypotheses:

▶ (H1): If Peter is old, then John is not the son of Peter
▶ (H2): If Peter is not old, then John is the son of Peter
▶ (H3): If John is Peter’s son then Mary is the sister of John

Conclusion (C): Mary is the sister of John, or Peter is old.

\[(p \Rightarrow \neg j) \land (\neg p \Rightarrow j) \land (j \Rightarrow m) \Rightarrow m \lor p\]

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Plan

Consequence

Important equivalences

Substitution and replacement

Normal forms

Conclusion
Plan

Consequence

Important equivalences

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Conclusion
Logical consequence (entailment)

Definition 1.2.24

A is a consequence of the set $\Gamma$ of hypotheses ($\Gamma \models A$) if every model of $\Gamma$ is a model of $A$.

Remark 1.2.26

We denote by $\models A$ the fact that $A$ is valid.
Indeed every truth assignment is a model for the empty set.
Example of a consequence

Example 1.2.28

\[ a \Rightarrow b, b \Rightarrow c \models a \Rightarrow c. \]
Example of a consequence

Example 1.2.28

\[ a \Rightarrow b, \ b \Rightarrow c \models a \Rightarrow c. \]

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ESSENTIAL property

Often used in exercises and during exams.

Property 1.2.27

Let $H_n = A_1 \land \ldots \land A_n$.

The following three formulations are equivalent:

1. $A_1, \ldots, A_n \models B$
2. $H_n \Rightarrow B$ is valid.
3. $H_n \land \neg B$ is unsatisfiable.

Proof.

Based on the truth tables of the connectives. We prove that $1 \Rightarrow 2$ then $2 \Rightarrow 3$ and $3 \Rightarrow 1$. □
Proof (1/3)

1 ⇒ 2: let us assume that $A_1, \ldots, A_n \models B$.

Let $\nu$ be a truth assignment:

- if $\nu$ is not a model for $A_1, \ldots, A_n$:
  
  for a certain $i$ we have $[A_i]_\nu = 0$, hence $[H_n]_\nu = 0$.
  
  Thus $[H_n \Rightarrow B]_\nu = 1$.

- if $\nu$ is a model for $A_1, \ldots, A_n$:
  
  then by hypothesis $\nu$ is a model for $B$ therefore $[B]_\nu = 1$.
  
  Thus $[H_n \Rightarrow B]_\nu = 1$.

Therefore $H_n \Rightarrow B$ is valid.
Proof (2/3)

- 2 ⇒ 3: let us assume that $H_n \Rightarrow B$ is valid.

For every truth assignment $\nu$:
- either $[H_n]_\nu = 0$,
- or $[H_n]_\nu = 1$ and $[B]_\nu = 1$.

However $[H_n \land \neg B]_\nu = \min([H_n]_\nu, [\neg B]_\nu) = \min([H_n]_\nu, 1 - [B]_\nu)$.

In both cases, we have $[H_n \land \neg B]_\nu = 0$.
Therefore $H_n \land \neg B$ is unsatisfiable.
Proof (3/3)

3 \Rightarrow 1: let us assume that \( H_n \land \neg B \) is unsatisfiable. Let us show that \( A_1, \ldots, A_n \models B \).

Let \( \nu \) be a truth assignment model of \( A_1, \ldots, A_n \):

- \( [H_n]_\nu = [A_1 \land \ldots \land A_n]_\nu = 1 \).
- According to our hypothesis \( [\neg B]_\nu = 0 \).

Hence, \( 1 - [B]_\nu = 0 \) so \( [B]_\nu = 1 \), i.e. \( \nu \) is a model for \( B \).

Exercise 7 shows why proving these 3 circular implications is sufficient.
## Instance of the property

**Example 1.2.28**

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Compactness

Theorem 1.2.30 Propositional compactness

A set of propositional formulae has a model if and only if every finite subset of it has a model.
Compactness

Theorem 1.2.30 Propositional compactness

A set of propositional formulae has a model if and only if every finite subset of it has a model.

This theorem may look trivial. However, the set of formulae may be infinite!

This result will be used at a later stage in the course (bases for automated theorem proving).
Plan

Consequence

Important equivalences

Substitution and replacement

Normal forms

Conclusion
How to prove that a formula is valid?
How to prove that a formula is valid?

- Truth table
  - Problem: for a formula having 100 variables, the truth table will contain $2^{100}$ lines (unable to be computed, even by a computer!).
Preamble

How to prove that a formula is valid?

- Truth table
  - Problem: for a formula having 100 variables, the truth table will contain $2^{100}$ lines (unable to be computed, even by a computer!).
  - Idea:
    - Simplify the formula using transformations
    - Then, study the simplified formula using truth tables or a logic reasoning
Disjunction

- **associative** $x \lor (y \lor z) \equiv (x \lor y) \lor z$
- **commutative** $x \lor y \equiv y \lor x$
- **idempotent** $x \lor x \equiv x$
Disjunction

- **associative** \( x \lor (y \lor z) \equiv (x \lor y) \lor z \)
- **commutative** \( x \lor y \equiv y \lor x \)
- **idempotent** \( x \lor x \equiv x \)

Ditto for conjunction.
Distributivity

- Conjunction distributes over disjunction
  \[ x \land (y \lor z) \equiv (x \land y) \lor (x \land z) \]
Distributivity

- Conjunction distributes over disjunction
  \[ x \land (y \lor z) \equiv (x \land y) \lor (x \land z) \]

- Disjunction distributes over conjunction
  \[ x \lor (y \land z) \equiv (x \lor y) \land (x \lor z) \]
Important equivalences

Neutrality and Absorption

- 0 is the neutral element for disjunction $0 \lor x \equiv x$
- 1 is the neutral element for conjunction $1 \land x \equiv x$
- 1 is the absorbing element for disjunction $1 \lor x \equiv 1$
- 0 is the absorbing element for conjunction $0 \land x \equiv 0$
Important equivalences

Negation

- Negation laws:
  - $x \land \neg x \equiv 0$
  - $x \lor \neg x \equiv 1$ (The law of excluded middle)
  - $\neg \neg x \equiv x$
  - $\neg 0 \equiv 1$
  - $\neg 1 \equiv 0$
De Morgan laws

\[ \neg (x \land y) \equiv \neg x \lor \neg y \]

\[ \neg (x \lor y) \equiv \neg x \land \neg y \]
Important equivalences

Simplification laws

Property 1.2.31

For every $x, y$ we have:

- $x \lor (x \land y) \equiv x$
- $x \land (x \lor y) \equiv x$
- $x \lor (\neg x \land y) \equiv x \lor y$
Plan

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Conclusion
Substitution

Definition 1.3.1 (uniform substitution)

A substitution $\sigma$ is a function mapping variables to formulae.
Substitution

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A substitution $\sigma$ is a function mapping variables to formulae.

$A\sigma = \text{the formula } A \text{ where all variables } x \text{ are replaced by the formula } \sigma(x)$. 
Substitution

Definition 1.3.1 (uniform substitution)

A substitution $\sigma$ is a function mapping variables to formulae.

$A\sigma$ = the formula $A$ where all variables $x$ are replaced by the formula $\sigma(x)$.

Example: $A = \neg(p \land q) \iff (\neg p \lor \neg q)$

- Let $\sigma$ the following substitution: $\sigma(p) = (a \lor b), \sigma(q) = (c \land d)$
- $A\sigma =$
Definition 1.3.1 (uniform substitution)

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Example: $A = \neg(p \land q) \iff (\neg p \lor \neg q)$

- Let $\sigma$ the following substitution: $\sigma(p) = (a \lor b)$, $\sigma(q) = (c \land d)$
- $A\sigma = \neg((a \lor b) \land (c \land d)) \iff (\neg(a \lor b) \lor \neg(c \land d))$
Finite support substitution

**Definition 1.3.2**

- The support of a substitution $\sigma$ is the set of variables $x$ such that $x\sigma \neq x$.
- A finite support substitution $\sigma$ is denoted $< x_1 := A_1, \ldots, x_n := A_n >$.
Finite support substitution

Definition 1.3.2

▶ The support of a substitution $\sigma$ is the set of variables $x$ such that $x\sigma \neq x$.

▶ A finite support substitution $\sigma$ is denoted $< x_1 := A_1, \ldots, x_n := A_n >$

Example 1.3.3

$A = x \lor x \land y \Rightarrow z \land y$ and $\sigma = < x := a \lor b, z := b \land c >$

$A\sigma =$

$A\sigma = (a \lor b) \lor (a \lor b) \land y \Rightarrow (b \land c) \land y$
Finite support substitution

Definition 1.3.2

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Example 1.3.3

$A = x \lor x \land y \Rightarrow z \land y$ and $\sigma = < x := a \lor b, z := b \land c >$

$A\sigma = (a \lor b) \lor (a \lor b) \land y \Rightarrow (b \land c) \land y$
Properties of substitutions

Property 1.3.4 (substitution and truth assignment)

Let $v$ be a truth assignment and $\sigma$ a substitution.
Let $w$ be the assignment $w : x \mapsto [\sigma(x)]_v$.
For any formula $A$, we have $[A\sigma]_v = [A]_w$. 

Example 1.3.5:

Let $A = x \lor y \lor d$
Let $\sigma = < x := a \lor b, y := b \land c >$
Let $v$ be $v(a) = 1, v(b) = 0, v(c) = 0, v(d) = 0$

$A\sigma = (a \lor b) \lor (b \land c) \lor d$

$[A\sigma]_v = 1 \lor 0 \lor 0 = 1$
$[A]_w = 1 \lor 0 \lor 0 = 1$
Properties of substitutions

Property 1.3.4 (substitution and truth assignment)

Let \( v \) be a truth assignment and \( \sigma \) a substitution.
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Properties of substitutions

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Example 1.3.5 :
Let \( A = x \lor y \lor d \)
Let \( \sigma = < x := a \lor b, y := b \land c > \)
Let \( v \) be \( v(a) = 1, v(b) = 0, v(c) = 0, v(d) = 0 \)

\[
A\sigma = (a \lor b) \lor (b \land c) \lor d
\]

\[
[A\sigma]_v = (1 \lor 0) \lor (0 \land 0) \lor 0
= 1 \lor 0 \lor 0 = 1
\]
Properties of substitutions

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Let $\sigma = < x := a \lor b, y := b \land c >$
Let $v$ be $v(a) = 1$, $v(b) = 0$, $v(c) = 0$, $v(d) = 0$

$A\sigma = (a \lor b) \lor (b \land c) \lor d$

$w(x) = [a \lor b]_v = 1 \lor 0 = 1$
$w(y) = [b \land c]_v = 0 \land 0 = 0$
$w(d) = [d]_v = 0$

$[A\sigma]_v = (1 \lor 0) \lor (0 \land 0) \lor 0$
$= 1 \lor 0 \lor 0 = 1$

$[A]_w = 1 \lor 0 \lor 0 = 1$
Proof by induction

Base case: $|A| = 0$

Two possible cases:
- If $A$ is $\top$ or $\bot$ then $A_σ = A$ and $[A]_ν$ does not depend on $ν$.
- If $A$ is a variable $x$, then by construction $[x_σ]_ν = w(x)$. 
Induction step case 1

**Hypothesis:** Assume the property holds for any formula of size less or equal to \( n \).
Let \( A \) be a formula of size \( n+1 \); there are two possible cases:

- **Case 1:** \( A = \neg B \) with \( |B| = n \).
  
  \[
  [A\sigma]_v = [\neg B\sigma]_v = [\neg (B\sigma)]_v = 1 - [B\sigma]_v \quad \text{and} \\
  [A]_w = [\neg B]_w = 1 - [B]_w.
  \]
  
  Since \( |B| = n \), by induction hypothesis \( [B\sigma]_v = [B]_w \)
  
  Hence, \( [A\sigma]_v = [A]_w \).
**Induction step case 2**

**Hypothesis:** Assume the property is true for any formula of size less or equal to $n$.
Let $A$ be a formula of size $n + 1$; there are two possible cases:

- **Case 2:** $A = (B \circ C)$ with $|B| < n + 1$ and $|C| < n + 1$.
  
  Then $[A\sigma]_v = [B\sigma \circ C\sigma]_v$
  and $[A]_w = [B \circ C]_w$

  By induction hypothesis $[B\sigma]_v = [B]_w$ and $[C\sigma]_v = [C]_w$.

Since the semantics for $\circ$ remain the same, $[A\sigma]_v = [A]_w$. 
Substitution of a valid formula

Theorem 1.3.6
If \(A\) is valid then \(A\sigma\) is valid for any \(\sigma\).

Proof.
Substitution and replacement

Substitution of a valid formula

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If $A$ is valid then $A\sigma$ is valid for any $\sigma$.

Proof.

Let $\nu$ be any truth assignment.
Substitution and replacement

Substitution of a valid formula

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If \( A \) is valid then \( A\sigma \) is valid for any \( \sigma \).

Proof.

Let \( \nu \) be any truth assignment.

According to property 1.3.4: \( [A\sigma]_\nu = [A]_w \) where \( w(x) = [\sigma(x)]_\nu \).
Theorem 1.3.6

If $A$ is valid then $A\sigma$ is valid for any $\sigma$.

Proof.

Let $\nu$ be any truth assignment.

According to property 1.3.4: $[A\sigma]_{\nu} = [A]_{w}$ where $w(x) = [\sigma(x)]_{\nu}$.

Since $A$ is valid, $[A]_{w} = 1$.

Consequently, $A\sigma$ equals 1 in every truth assignment, therefore $A\sigma$ is a valid formula.
Example 1.3.7

Let $A$ be the formula $\neg(p \land q) \iff (\neg p \lor \neg q)$. This formula is valid, it is an important equivalence. Let $\sigma$ the following substitution:

$<p := (a \lor b), q := (c \land d)>$. The formula
Examples

Example 1.3.7

Let $A$ be the formula $\neg(p \land q) \iff (\neg p \lor \neg q)$. This formula is valid, it is an important equivalence. Let $\sigma$ the following substitution: $< p := (a \lor b), q := (c \land d) >$. The formula

$$A\sigma = \neg((a \lor b) \land (c \land d)) \iff (\neg(a \lor b) \lor \neg(c \land d))$$

is also valid.
Replacement

Definition 1.3.8

The formula $D$ is obtained by replacing certain occurrences of $A$ by $B$ in $C$ if:

- $C$ can be written $E < x := A >$
- $D$ can be written $E < x := B >$

for some formula $E$. 
Examples

Example 1.3.9

Consider the formula $C = ((a \Rightarrow b) \lor \neg(a \Rightarrow b))$.

► The formula obtained by replacing all occurrences of $(a \Rightarrow b)$ by $(a \land b)$ is
Examples

Example 1.3.9

Consider the formula \( C = ((a \Rightarrow b) \lor \neg(a \Rightarrow b)) \).

The formula obtained by replacing all occurrences of \( (a \Rightarrow b) \) by \( (a \land b) \) is

\[
D = ((a \land b) \lor \neg(a \land b))
\]

using \( E = (x \lor \neg x) \).
### Examples

**Example 1.3.9**

Consider the formula $C = ((a \Rightarrow b) \lor \neg(a \Rightarrow b))$.

- The formula obtained by replacing all occurrences of $(a \Rightarrow b)$ by $(a \land b)$ is

  $D = ((a \land b) \lor \neg(a \land b))$

  using $E = (x \lor \neg x)$.

- The formula obtained by replacing the *first* occurrence of $(a \Rightarrow b)$ by $(a \land b)$ is
Examples

Example 1.3.9

Consider the formula $C = ((a \implies b) \lor \neg(a \implies b))$.

- The formula obtained by replacing all occurrences of $(a \implies b)$ by $(a \land b)$ is
  
  $D = ((a \land b) \lor \neg(a \land b))$

  using $E = (x \lor \neg x)$.

- The formula obtained by replacing the first occurrence of $(a \implies b)$ by $(a \land b)$ is
  
  $D = ((a \land b) \lor \neg(a \implies b))$

  using $E = (x \lor \neg(a \implies b))$. 
Properties of replacement

Theorem 1.3.10 (replacement of equivalents)

If $D$ is obtained by replacing, in $C$, some occurrences of $A$ by $B$, then $(A \iff B) \models (C \iff D)$.
Properties of replacement

Theorem 1.3.10 (replacement of equivalents)

If \( D \) is obtained by replacing, in \( C \), some occurrences of \( A \) by \( B \), then
\[
(A \iff B) \models (C \iff D).
\]

Proof.

By definition, \( C = E < x := A > \) et \( D = E < x := B > \).
Assume that \( [A]_\nu = [B]_\nu \), then \( \nu \) is the same for both substitutions.
Therefore \( [C]_\nu = [D]_\nu \) : the assignment \( \nu \) is a model of \( (C \iff D) \). \( \square \)
Properties of replacement

Theorem 1.3.10 (replacement of equivalents)

If $D$ is obtained by replacing, in $C$, some occurrences of $A$ by $B$, then $(A \Leftrightarrow B) \vdash (C \Leftrightarrow D)$.

Proof.

By definition, $C = E < x := A >$ et $D = E < x := B >$. Assume that $[A]_\nu = [B]_\nu$, then $w$ is the same for both substitutions. Therefore $[C]_\nu = [D]_\nu :$ the assignment $\nu$ is a model of $(C \Leftrightarrow D)$. 

Example 1.3.12: $p \Leftrightarrow q \vdash (p \lor (p \Rightarrow r)) \Leftrightarrow (p \lor (q \Rightarrow r))$. 

Corollary 1.3.11

Let $D$ be obtained by replacing, in $C$, one occurrence of $A$ by $B$. If $A \equiv B$ then $C \equiv D$. 

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Properties of replacement

Theorem 1.3.10 (replacement of equivalents)

If $D$ is obtained by replacing, in $C$, some occurrences of $A$ by $B$, then $(A \iff B) \models (C \iff D)$.

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By definition, $C = E < x := A >$ et $D = E < x := B >$.
Assume that $[A]_\nu = [B]_\nu$, then $w$ is the same for both substitutions. Therefore $[C]_\nu = [D]_\nu$ : the assignment $\nu$ is a model of $(C \iff D)$. \qed

Example 1.3.12: $p \iff q \models (p \lor (\boxed{p} \Rightarrow r)) \iff (p \lor (\boxed{q} \Rightarrow r))$.

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Let $D$ be obtained by replacing, in $C$, one occurrence of $A$ by $B$. If $A \equiv B$ then $C \equiv D$. 
Plan

Consequence

Important equivalences

Substitution and replacement

Normal forms

Conclusion
Definitions

Definition 1.4.1

- A literal is a variable or its negation.
Definitions

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- A **literal** is a variable or its negation.
- A **monomial** is a conjunction of literals (special cases 0 and 1).
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Example 1.4.2

▶ $x, y, \neg z$ are literals.
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- The monomial $x \land \neg y \land z \land \neg x$ contains $x$ and $\neg x$: its value is 0.
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 Definitions

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- $x, y, \neg z$ are literals.
- $x \land \neg y \land z$ is a monomial
- The monomial $x \land \neg y \land z \land \neg x$ contains $x$ and $\neg x$: its value is 0.
- $x \lor \neg y \lor z$ is a clause
- The clause $x \lor \neg y \lor z \lor \neg x$ contains $x$ and $\neg x$: its value is 1.
Normal forms

Definition 1.4.3

A formula is in **normal form** if it only contains the operators $\land, \lor, \neg$ and the negations are only applied to **variables**.
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**Example 1.4.4**

The formula $\neg a \lor b$ is in normal form

$a \implies b$ is not in normal form, even if it is equivalent to the first one.
Definition 1.4.3

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Example 1.4.4

The formula $\neg a \lor b$ is in normal form
$a \Rightarrow b$ is not in normal form, even if it is equivalent to the first one.

Every formula admits an equivalent normal form.
Computing a normal form

1. Equivalence elimination

2. Implication elimination

3. Shifting negations towards variables
Computing a normal form

1. **Equivalence elimination**
   Replace any occurrence of \( A \iff B \) by
   
   (a) \((\neg A \lor B) \land (\neg B \lor A)\)
   
   OR
   
   (b) \((A \land B) \lor (\neg A \land \neg B)\)

2. **Implication elimination**

3. **Shifting negations towards variables**
Computing a normal form

1. **Equivalence elimination**
   Replace any occurrence of $A \Leftrightarrow B$ by
   (a) $(\neg A \lor B) \land (\neg B \lor A)$
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   OR
   
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2. **Implication elimination**
   Replace any occurrence of $A \implies B$ by $\neg A \lor B$

3. **Shifting negations towards variables**
   Replace any occurrence of
   
   (a) $\neg \neg A$ by $A$
   
   (b) $\neg (A \lor B)$ by $\neg A \land \neg B$
   
   (c) $\neg (A \land B)$ by $\neg A \lor \neg B$
Remark 1.4.5: simplifications

Simplify as soon as possible:

1. Replace \( \neg (A \Rightarrow B) \) by \( A \land \neg B \).
Remark 1.4.5: simplifications

Simplify as soon as possible:

1. Replace $\neg(A \Rightarrow B)$ by $A \land \neg B$.
2. Replacing a conjunction by 0 if it contains a formula and its negation
3. Replace a disjunction by 1 if it contains a formula and its negation
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1. Replace $\neg(A \Rightarrow B)$ by $A \land \neg B$.
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4. Apply :
   - Idempotence of $\land$ and $\lor$
   - Neutrality and absorption of 0 and 1
   - Replace $\neg 1$ by 0 and vice versa.
Remark 1.4.5: simplifications

Simplify as soon as possible:

1. Replace \( \neg(A \Rightarrow B) \) by \( A \land \neg B \).
2. Replacing a conjunction by 0 if it contains a formula and its negation
3. Replace a disjunction by 1 if it contains a formula and its negation
4. Apply :
   - Idempotence of \( \land \) and \( \lor \)
   - Neutrality and absorption of 0 and 1
   - Replace \( \neg 1 \) by 0 and vice versa.
5. Apply the simplifications:
   - \( x \lor (x \land y) \equiv x \),
   - \( x \land (x \lor y) \equiv x \),
   - \( x \lor (\neg x \land y) \equiv x \lor y \)
Disjunctive normal form (DNF)

Definition 1.4.6

A formula is in disjunctive normal form (DNF) if and only if it is a disjunction (sum) of monomials.

Method: distribute the conjunctions over the disjunctions

\[ x \land (y \lor z) \equiv (x \land y) \lor (x \land z) \]
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\[(x \land y) \lor (\neg x \land \neg y \land z)\] is a DNF, which has two main models:

- \( x \mapsto 1, y \mapsto 1 \)
- \( x \mapsto 0, y \mapsto 0, z \mapsto 1 \)
Conjunctive normal form (CNF)

Definition 1.4.11

A formula is a conjunctive normal form (CNF) if and only if it is a conjunction (product) of clauses.

Apply the (unusual) distributivity of disjunction over conjunction:

\[ A \lor (B \land C) \equiv (A \lor B) \land (A \lor C) \]
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**Example 1.4.12**

\[(x \lor y) \land (\neg x \lor \neg y \lor z)\] is a CNF, which has two counter-models.
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The interest of a CNF is to highlight the counter-models.

**Example 1.4.12**

\[(x \lor y) \land (\neg x \lor \neg y \lor z)\]

is a CNF, which has two counter-models.

- \(x \mapsto 0, y \mapsto 0\)
- \(x \mapsto 1, y \mapsto 1, z \mapsto 0\).
Examples 1.4.8 and 1.4.13

Transformation in DNF of the following:

\[(a \lor b) \land (c \lor d \lor e) \equiv \]
Examples 1.4.8 and 1.4.13

Transformation in DNF of the following:

$$(a \lor b) \land (c \lor d \lor e) \equiv (a \land c) \lor (a \land d) \lor (a \land e) \lor (b \land c) \lor (b \land d) \lor (b \land e).$$
Examples 1.4.8 and 1.4.13

Transformation in DNF of the following:

$$(a \lor b) \land (c \lor d \lor e) \equiv \hfill (a \land c) \lor (a \land d) \lor (a \land e) \lor (b \land c) \lor (b \land d) \lor (b \land e).$$

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Examples 1.4.8 and 1.4.13

Transformation in **DNF** of the following:

\[(a \lor b) \land (c \lor d \lor e) \equiv (a \land c) \lor (a \land d) \lor (a \land e) \lor (b \land c) \lor (b \land d) \lor (b \land e).\]

Transformation in **CNF** of the following:

\[(a \land b) \lor (c \land d \land e) \equiv (a \lor c) \land (a \lor d) \land (a \lor e) \land (b \lor c) \land (b \lor d) \land (b \lor e).\]
Another use of DNFs

Transforming a formula into a disjunction of monomials also allows us to determine whether the formula is valid or not.
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Let $A$ be a formula whose validity we wish to check:
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Transforming a formula into a disjunction of monomials also allows us to determine whether the formula is valid or not.

Let $A$ be a formula whose validity we wish to check:

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- If $B = 0$ then $\neg A = 0$, hence $A = 1$, that is, $A$ is valid
Another use of DNFs

Transforming a formula into a disjunction of monomials also allows us to determine whether the formula is valid or not.

Let $A$ be a formula whose validity we wish to check:

We transform $\neg A$ in an equivalent disjunction of monomials $B$:

- If $B = 0$ then $\neg A = 0$, hence $A = 1$, that is, $A$ is valid
- Otherwise $B$ is equal to a disjunction of nonzero monomials equivalent to $\neg A$, which give us models of $\neg A$, hence counter-models of $A$. 
Example 1.4.9

Let $A = (p \Rightarrow (q \Rightarrow r)) \Rightarrow (p \land q \Rightarrow r)$

Determine whether $A$ is valid.

$\neg A$
Example 1.4.9

Let \( A = (p \Rightarrow (q \Rightarrow r)) \Rightarrow (p \land q \Rightarrow r) \)

Determine whether \( A \) is valid.

\[
\neg A \\
\equiv (p \Rightarrow (q \Rightarrow r)) \land \neg (p \land q \Rightarrow r) \quad \text{since } \neg (B \Rightarrow C) \equiv B \land \neg C
\]
Example 1.4.9

Let \( A = (p \Rightarrow (q \Rightarrow r)) \Rightarrow (p \land q \Rightarrow r) \)

Determine whether \( A \) is valid.

\[
\neg A \\
\equiv (p \Rightarrow (q \Rightarrow r)) \land \neg (p \land q \Rightarrow r) \\
\equiv (\neg p \lor \neg q \lor r) \land \neg (p \land q \Rightarrow r)
\]

since \( \neg (B \Rightarrow C) \equiv B \land \neg C \)

eliminating two \( \Rightarrow \)
Example 1.4.9

Let \( A = (p \Rightarrow (q \Rightarrow r)) \Rightarrow (p \land q \Rightarrow r) \)

Determine whether \( A \) is valid.

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\neg A \\
\equiv (p \Rightarrow (q \Rightarrow r)) \land \neg (p \land q \Rightarrow r) \\
\equiv (\neg p \lor \neg q \lor r) \land \neg (p \land q \Rightarrow r) \\
\equiv (\neg p \lor \neg q \lor r) \land (p \land q \land \neg r) \\
\equiv (\neg (B \Rightarrow C)) \Leftrightarrow B \land \neg C
\]
Example 1.4.9

Let $A = (p \Rightarrow (q \Rightarrow r)) \Rightarrow (p \land q \Rightarrow r)$

Determine whether $A$ is valid.

\[
\neg A \\
\equiv (p \Rightarrow (q \Rightarrow r)) \land \neg (p \land q \Rightarrow r) \quad \text{since } \neg (B \Rightarrow C) \equiv B \land \neg C \\
\equiv (\neg p \lor \neg q \lor r) \land \neg (p \land q \Rightarrow r) \quad \text{eliminating two } \Rightarrow \\
\equiv (\neg p \lor \neg q \lor r) \land (p \land q \land \neg r) \quad \text{since } \neg (B \Rightarrow C) \equiv B \land \neg C \\
\equiv (\neg q \lor r) \land p \land q \land \neg r \quad \text{simplification } x \land (\neg x \lor y)
\]
Example 1.4.9

Let $A = (p \Rightarrow (q \Rightarrow r)) \Rightarrow (p \land q \Rightarrow r)$

Determine whether $A$ is valid.

\[
\neg A \\
\equiv (p \Rightarrow (q \Rightarrow r)) \land \neg(p \land q \Rightarrow r) \\
\equiv (\neg p \lor \neg q \lor r) \land \neg(p \land q \Rightarrow r) \\
\equiv (\neg p \lor \neg q \lor r) \land (p \land q \land \neg r) \\
\equiv (\neg q \lor r) \land p \land q \land \neg r \\
\equiv (r) \land p \land q \land \neg r
\]

since $\neg(B \Rightarrow C) \equiv B \land \neg C$

eliminating two $\Rightarrow$

simplification $x \land (\neg x \lor y)$

simplification $x \land (\neg x \lor y)$

Hence $\neg A = 0$ and $A = 1$, that is $A$ is valid.
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Let \( A = (p \Rightarrow (q \Rightarrow r)) \Rightarrow (p \land q \Rightarrow r) \)

Determine whether \( A \) is valid.

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\neg A \\
\equiv (p \Rightarrow (q \Rightarrow r)) \land \neg(p \land q \Rightarrow r) \quad \text{since } \neg(B \Rightarrow C) \equiv B \land \neg C \\
\equiv (\neg p \lor \neg q \lor r) \land \neg(p \land q \Rightarrow r) \quad \text{eliminating two } \Rightarrow \\
\equiv (\neg p \lor \neg q \lor r) \land (p \land q \land \neg r) \quad \text{since } \neg(B \Rightarrow C) \equiv B \land \neg C \\
\equiv (\neg q \lor r) \land p \land q \land \neg r \quad \text{simplification } x \land (\neg x \lor y) \\
\equiv (r) \land p \land q \land \neg r \quad \text{simplification } x \land (\neg x \lor y) \\
= 0 \\
\]

Hence \( \neg A = 0 \) and \( A = 1 \), that is \( A \) is valid.
Example 1.4.10

Let $A = (a \Rightarrow b) \land c \lor (a \land d)$.

Determine whether $A$ is valid.

$\neg A$
Example 1.4.10

Let \( A = (a \implies b) \land c \lor (a \land d) \).

Determine whether \( A \) is valid.

\[
\neg A = \neg ((a \implies b) \land c) \land \neg (a \land d) \quad \text{(de Morgan)}
\]

We obtain 3 models of \( \neg A \): 
\( (a \mapsto 1, b \mapsto 0, d \mapsto 0) \), 
\( (a \mapsto 0, c \mapsto 0) \), 
\( (c \mapsto 0, d \mapsto 0) \).

That is, counter-models of \( A \).

Hence \( A \) is not valid.
Example 1.4.10

Let $A = (a \Rightarrow b) \land c \lor (a \land d)$.

Determine whether $A$ is valid.

\[
\neg A \\
\equiv \neg((a \Rightarrow b) \land c) \land \neg(a \land d) \quad \text{(de Morgan)} \\
\equiv (\neg(a \Rightarrow b) \lor \neg c) \land (\neg a \lor \neg d) \quad \text{(de Morgan)}
\]
Example 1.4.10

Let $A = (a \Rightarrow b) \land c \lor (a \land d)$.

Determine whether $A$ is valid.

\[
\neg A \\
\equiv \neg((a \Rightarrow b) \land c) \land \neg(a \land d) \quad \text{(de Morgan)} \\
\equiv ((\neg(a \Rightarrow b) \lor \neg c) \land (\neg a \lor \neg d)) \quad \text{(de Morgan)} \\
\equiv ((a \land \neg b) \lor \neg c) \land (\neg a \lor \neg d) \quad \text{elimination of the implication}
\]
Example 1.4.10

Let \( A = (a \Rightarrow b) \land c \lor (a \land d). \)

Determine whether \( A \) is valid.

\[
\neg A \\
\equiv \neg((a \Rightarrow b) \land c) \land \neg(a \land d) \quad \text{(de Morgan)} \\
\equiv (\neg(a \Rightarrow b) \lor \neg c) \land (\neg a \lor \neg d) \quad \text{(de Morgan)} \\
\equiv ((a \land \neg b) \lor \neg c) \land (\neg a \lor \neg d) \quad \text{elimination of the implication} \\
\equiv (a \land \neg b \land \neg a) \lor (a \land \neg b \land \neg d) \lor (\neg c \land \neg a) \lor (\neg c \land \neg d) \quad \text{distributivity of} \lor \text{ over} \land
\]
Example 1.4.10

Let \( A = (a \Rightarrow b) \land c \lor (a \land d) \).

Determine whether \( A \) is valid.

\[
\neg A \\
\equiv \neg((a \Rightarrow b) \land c) \land \neg(a \land d) \quad \text{(de Morgan)} \\
\equiv ((a \land \neg b) \lor \neg c) \land (\neg a \lor \neg d) \quad \text{(de Morgan)} \\
\equiv ((a \land \neg b) \lor \neg c) \land (\neg a \lor \neg d) \quad \text{elimination of the implication} \\
\equiv (a \land \neg b \land \neg a) \lor (a \land \neg b \land \neg d) \lor (\neg c \land \neg a) \lor (\neg c \land \neg d) \quad \text{distributivity of } \lor \text{ over } \land \\
\equiv (a \land \neg b \land \neg d) \lor (\neg c \land \neg a) \lor (\neg c \land \neg d) \quad 1\text{st monomial contradictory}
\]
Example 1.4.10

Let \( A = (a \Rightarrow b) \land c \lor (a \land d) \).

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\equiv ((a \land \neg b) \lor \neg c) \land (\neg a \lor \neg d) \quad \text{elimination of the implication} \\
\equiv (a \land \neg b \land \neg a) \lor (a \land \neg b \land \neg d) \lor (\neg c \land \neg a) \lor (\neg c \land \neg d) \quad \text{distributivity of } \lor \text{ over } \land \\
\equiv (a \land \neg b \land \neg d) \lor (\neg c \land \neg a) \lor (\neg c \land \neg d) \quad \text{1st monomial contradictory}
\]

We obtain 3 models of \( \neg A \): \((a \mapsto 1, b \mapsto 0, d \mapsto 0)\), \((a \mapsto 0, c \mapsto 0)\), \((c \mapsto 0, d \mapsto 0)\).
That is, counter-models of \( A \).
Hence \( A \) is not valid.
Plan

Consequence

Important equivalences

Substitution and replacement

Normal forms

Conclusion
Today

- **Substitutions** allow us to **deduce the validity** of a formula from another.
- **Replacements** allow us to change part of a formula **without changing its meaning** and thus allow us to compute a simpler equivalent formula.
- Every formula admits **normal forms** which allow to **highlight its models** and counter-models.
Next course

- Boolean algebra
- Boolean functions
- Resolution

Prove our example by formula simplification

\[(p \Rightarrow \neg j) \land (\neg p \Rightarrow j) \land (j \Rightarrow m) \Rightarrow m \lor p\]