Propositional Resolution

Second Part: Algorithms

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Proof by resolution of our running example

- (H1): $p \Rightarrow \neg j \equiv \neg p \lor \neg j$
- (H2): $\neg p \Rightarrow j \equiv p \lor j$
- (H3): $j \Rightarrow m \equiv \neg j \lor m$
- (C): $m \lor p$

To prove: $H1, H2, H3 \vdash C$
Proof by resolution of our running example

- (H1) : \( p \Rightarrow \neg j \equiv \neg p \lor \neg j \)
- (H2) : \( \neg p \Rightarrow j \equiv p \lor j \)
- (H3) : \( j \Rightarrow m \equiv \neg j \lor m \)
- (C) : \( m \lor p \)

To prove: \( H1, H2, H3 \vdash C \)

- \( \neg C \equiv \neg m \land \neg p \)

Clauses: \( \{ \neg p \lor \neg j, p \lor j, \neg j \lor m, \neg m, \neg p \} \)
Proof by resolution of our running example

\[\begin{align*}
\text{(H1)} &: p \Rightarrow \neg j \equiv \neg p \lor \neg j \\
\text{(H2)} &: \neg p \Rightarrow j \equiv p \lor j \\
\text{(H3)} &: j \Rightarrow m \equiv \neg j \lor m \\
\text{(C)} &: m \lor p
\end{align*}\]

To prove: $H1, H2, H3 \vdash C$

\[\neg C \equiv \neg m \land \neg p\]

Clauses: $\{\neg p \lor \neg j, p \lor j, \neg j \lor m, \neg m, \neg p\}$

\[\begin{align*}
p \lor j & \quad \neg j \lor m \\
\hline
p \lor m & \quad \neg m \\
\hline
p \lor m & \quad \neg m \\
\hline
p & \quad \neg p \\
\hline
\bot & \quad \bot
\end{align*}\]
Proof by resolution of our running example

- (H1): $p \Rightarrow \neg j \equiv \neg p \lor \neg j$
- (H2): $\neg p \Rightarrow j \equiv p \lor j$
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- (C): $m \lor p$

To prove: $H1, H2, H3 \vdash C$

- $\neg C \equiv \neg m \land \neg p$

Clauses: \{\neg p \lor \neg j, p \lor j, \neg j \lor m, \neg m, \neg p\}

\[
\begin{align*}
\frac{p \lor j \hspace{1cm} \neg j \lor m}{p \lor m} \quad \frac{p \lor j \hspace{1cm} \neg p}{\neg m} & \quad \frac{p \lor j \hspace{1cm} \neg p}{\neg m} \\
\hline
\frac{\neg m}{p} \quad \frac{\neg m}{p} & \quad \frac{\neg m}{p} \\
\hline
\frac{\bot}{\bot} & \quad \frac{\bot}{\bot}
\end{align*}
\]

OR
Previous lectures

- Language: Propositional Logic
- Semantics: Truth Tables, Boolean Algebras
- Systems of Deduction: Resolution
Previous lectures

- Language: Propositional Logic
- Semantics: Truth Tables, Boolean Algebras
- Systems of Deduction: Resolution

1. $\Gamma \vdash B$

   $B$ is *deduced* from $\Gamma$: there is a formal proof (by resolution) of $B$ starting from $\Gamma$. 

2. $\Gamma \models B$

   $B$ is a *consequence* of $\Gamma$: every model of $\Gamma$ is also a model of $B$. 

Today: Correctness

(1) $\Rightarrow$ (2)

Today: Completeness

(2) $\Rightarrow$ (1)
Previous lectures

- Language: Propositional Logic
- Semantics: Truth Tables, Boolean Algebras
- Systems of Deduction: Resolution

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Overview

Correctness of deduction

Completeness of deduction

Introduction to Resolution Algorithms

A Deductive Method: Complete Strategy

A SAT Method: the Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

Conclusion
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Completeness of deduction

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Conclusion
Definition

The **correctness** of a deductive system states that all the statements which can be proved in the systems are true (in the sense of our truth-value semantics).

If it is 'proved', then it is 'true'.
Correctness of the resolution rule

Theorem 2.1.15

If $C$ is a resolvent of $A$ and $B$ then $A, B \models C$.

Proof.
Correctness of the resolution rule

**Theorem 2.1.15**

If $C$ is a resolvent of $A$ and $B$ then $A, B \models C$.

**Proof.**

If $C$ is a resolvent of $A$ and $B$, then there is a literal $L$ such that $L \in A, L^c \in B$, and $C = (A - \{L\}) \cup (B - \{L^c\})$.
Correctness of the resolution rule

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If $C$ is a resolvent of $A$ and $B$, then there is a literal $L$ such that $L \in A, L^c \in B$, and $C = (A - \{L\}) \cup (B - \{L^c\})$.
Let $\nu$ be an assignment such that $[A]_\nu = 1$ and $[B]_\nu = 1$: let us show that $[C]_\nu = 1$. 

$\square$
Correctness of the resolution rule

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If \( C \) is a resolvent of \( A \) and \( B \) then \( A, B \models C \).

Proof.

If \( C \) is a resolvent of \( A \) and \( B \), then there is a literal \( L \) such that \( L \in A, L^c \in B \), and \( C = (A - \{L\}) \cup (B - \{L^c\}) \).

Let \( v \) be an assignment such that \( [A]_v = 1 \) and \( [B]_v = 1 \): let us show that \( [C]_v = 1 \).

- Suppose that \( [L]_v = 1 \).
- Suppose that \( [L^c]_v = 1 \).
Correctness of the resolution rule

Theorem 2.1.15
If \(C\) is a resolvent of \(A\) and \(B\) then \(A, B \models C\).

Proof.
If \(C\) is a resolvent of \(A\) and \(B\), then there is a literal \(L\) such that \(L \in A, L^c \in B\), and \(C = (A - \{L\}) \cup (B - \{L^c\})\).
Let \(\nu\) be an assignment such that \([A]_{\nu} = 1\) and \([B]_{\nu} = 1\): let us show that \([C]_{\nu} = 1\).

- Suppose that \([L]_{\nu} = 1\). Therefore \([L^c]_{\nu} = 0\).
  Since \([B]_{\nu} = 1\), \(\nu\) is a model of a literal of \((B - \{L^c\})\). Hence \([C]_{\nu} = 1\).
- Suppose that \([L^c]_{\nu} = 1\).
Correctness of the resolution rule

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If $C$ is a resolvent of $A$ and $B$ then $A, B \models C$.

Proof.

If $C$ is a resolvent of $A$ and $B$, then there is a literal $L$ such that $L \in A, L^c \in B$, and $C = (A - \{L\}) \cup (B - \{L^c\})$.

Let $\nu$ be an assignment such that $[A]_{\nu} = 1$ and $[B]_{\nu} = 1$: let us show that $[C]_{\nu} = 1$.

- Suppose that $[L]_{\nu} = 1$. Therefore $[L^c]_{\nu} = 0$.
  Since $[B]_{\nu} = 1$, $\nu$ is a model of a literal of $(B - \{L^c\})$. Hence $[C]_{\nu} = 1$.

- Suppose that $[L^c]_{\nu} = 1$. Therefore $[L]_{\nu} = 0$.
  Since $[A]_{\nu} = 1$, $\nu$ is a model of $(A - \{L\})$. Hence $[C]_{\nu} = 1$. 

□
Correctness of the resolution rule

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If $C$ is a resolvent of $A$ and $B$ then $A, B \models C$.

Proof.
If $C$ is a resolvent of $A$ and $B$, then there is a literal $L$ such that $L \in A, L^c \in B$, and $C = (A - \{L\}) \cup (B - \{L^c\})$.

Let $\nu$ be an assignment such that $[A]_{\nu} = 1$ and $[B]_{\nu} = 1$: let us show that $[C]_{\nu} = 1$.

- Suppose that $[L]_{\nu} = 1$. Therefore $[L^c]_{\nu} = 0$.
  Since $[B]_{\nu} = 1$, $\nu$ is a model of a literal of $(B - \{L^c\})$. Hence $[C]_{\nu} = 1$.

- Suppose that $[L^c]_{\nu} = 1$. Therefore $[L]_{\nu} = 0$.
  Since $[A]_{\nu} = 1$, $\nu$ is a model of $(A - \{L\})$. Hence $[C]_{\nu} = 1$.

Since every truth assignment is either model of $L$ or $L^c$, $\nu$ is a model of $C$. □
Correctness of deduction

Theorem 2.1.16 (by induction on the length of the derivation)

Let \( \Gamma \) be a set of clauses and \( C \) a clause. If \( \Gamma \vdash C \) then \( \Gamma \models C \).

Proof.

Suppose that there is a proof \( P \) of \( C \) starting from \( \Gamma \).
Suppose that for any proof of \( \Gamma \vdash D \) shorter than \( P \), we have \( \Gamma \models D \).
Let us show that \( \Gamma \models C \). There are two possible cases:

1. \( C \) is a member of \( \Gamma \), in this case \( \Gamma \models C \).
2. \( \Gamma \vdash A \), \( \Gamma \vdash B \) (with a shorter proof) and \( A \lor B \vdash C \).
   By induction hypothesis: \( \Gamma \models A \) and \( \Gamma \models B \).
   By correctness of the resolution rule: \( A \lor B \models C \). Hence \( \Gamma \models C \).
Correctness of deduction

Theorem 2.1.16 (by induction on the length of the derivation)

Let $\Gamma$ be a set of clauses and $C$ a clause. If $\Gamma \vdash C$ then $\Gamma \models C$.

Proof.

Suppose that there is a proof $P$ of $C$ starting from $\Gamma$.
Suppose that for any proof of $\Gamma \vdash D$ shorter than $P$, we have $\Gamma \models D$.
Let us show that $\Gamma \models C$. There are two possible cases:

1. $C$ is a member of $\Gamma$, in this case $\Gamma \models C$. 
Correctness of deduction

Theorem 2.1.16 (by induction on the length of the derivation)

Let $\Gamma$ be a set of clauses and $C$ a clause. If $\Gamma \vdash C$ then $\Gamma \models C$.

Proof.

Suppose that there is a proof $P$ of $C$ starting from $\Gamma$.
Suppose that for any proof of $\Gamma \vdash D$ shorter than $P$, we have $\Gamma \models D$.
Let us show that $\Gamma \models C$. There are two possible cases:

1. $C$ is a member of $\Gamma$, in this case $\Gamma \models C$.
2. $\Gamma \vdash A$, $\Gamma \vdash B$ (with a shorter proof) and

\[
\frac{A \quad B}{C}
\]

By induction hypothesis: $\Gamma \models A$ and $\Gamma \models B$.
By correctness of the resolution rule: $A, B \models C$. Hence $\Gamma \models C$. 
Overview

Correctness of deduction

Completeness of deduction

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Conclusion
Definition

Completeness for refutation is the property:

If $\Gamma \models \bot$ then $\Gamma \vdash \bot$

We will prove this result for a finite $\Gamma$, by induction on the number of variables in $\Gamma$. 
Definition 2.1.18

Let $\Gamma$ be a set of clauses and $L$ a literal.

$\Gamma[L := 1]$ is obtained by:
- deleting the clauses containing $L$
- removing $L^c$ from the other clauses.

$\Gamma[L := 0]$ is similarly defined by switching the roles of $L$ and $L^c$.

Remark: the number of variables in $\Gamma$ has been decreased.
Examples

Example 2.1.19

Let \( \Gamma \) be the set of clauses \( \overline{p} + q, \overline{q} + r, p + q, p + r \). We have:

\[
\Gamma[p := 1] = \{ q, q + r \}
\]

\[
\Gamma[p := 0] = \{ q + r, q, r \}
\]

Notice that:

\[
(1 + q)(q + r)(1 + q)(1 + r) \equiv q(q + r) = \Gamma[p := 1]
\]

\[
(0 + q)(q + r)(0 + q)(0 + r) \equiv q + r = \Gamma[p := 0]
\]
Examples

Example 2.1.19

Let $\Gamma$ be the set of clauses $\overline{p} + q$, $\overline{q} + r$, $p + q$, $p + r$. We have:

- $\Gamma[p := 1] =$
  $$\{q, \overline{q} + r\}.$$

- $\Gamma[p := 0] =$
## Examples

**Example 2.1.19**

Let $\Gamma$ be the set of clauses $\overline{p} + q$, $\overline{q} + r$, $p + q$, $p + r$. We have:

- $\Gamma[p := 1] = \{q, \overline{q} + r\}$.
- $\Gamma[p := 0] = \{\overline{q} + r, q, r\}$. 

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## Examples

### Example 2.1.19

Let $\Gamma$ be the set of clauses $\overline{p} + q$, $\overline{q} + r$, $p + q$, $p + r$. We have:

- $\Gamma[p := 1] = \{q, \overline{q} + r\}$.
- $\Gamma[p := 0] = \{\overline{q} + r, q, r\}$.

Notice that:

- $(\overline{1} + q)(\overline{q} + r)(1 + q)(1 + r) \equiv q(\overline{q} + r) = \Gamma[p := 1]$.
- $(\overline{0} + q)(\overline{q} + r)(0 + q)(0 + r) \equiv (\overline{q} + r)qr = \Gamma[p := 0]$. 

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Property of $\Gamma[L := ...]$

Property 2.1.21

$\Gamma$ has a model if and only if $\Gamma[L := 1]$ or $\Gamma[L := 0]$ has a model.

Proof.
Property of $\Gamma[L := \ldots]$ 

Property 2.1.21

$\Gamma$ has a model if and only if $\Gamma[L := 1]$ or $\Gamma[L := 0]$ has a model.

Proof.

$\Rightarrow$ If $\nu$ is a model of $\Gamma$ then $\nu$ is a model of either $\Gamma[L := 0]$ (if $[L]_{\nu'} = 0$) or $\Gamma[L := 1]$ (if $[L]_{\nu'} = 1$).
Property of $\Gamma[L := \ldots]$ 

Property 2.1.21

$\Gamma$ has a model if and only if $\Gamma[L := 1]$ or $\Gamma[L := 0]$ has a model.

Proof.

⇒ If $\nu$ is a model of $\Gamma$ then $\nu$ is a model of either $\Gamma[L := 0]$ (if $[L]_{\nu'} = 0$) or $\Gamma[L := 1]$ (if $[L]_{\nu'} = 1$).

⇐ If $\nu$ is a model of $\Gamma[L := i]$ then we can build a model of $\Gamma$ (by taking $[L]_{\nu'} = i$).
Lemma 2.1.22

Let $\Gamma$ a set of clauses, $C$ a clause and $L$ a literal.
If $\Gamma[L := 1] \vdash C$ then $\Gamma \vdash C$ or $\Gamma \vdash C + L^c$.

Proof.

Idea: we put back $L^c$ in the clauses where it was removed.

- If $C \in \Gamma[L := 1]$:

- If $C$ is a resolvent of $A$ and $B$:
Lemma 2.1.22

Let $\Gamma$ a set of clauses, $C$ a clause and $L$ a literal.
If $\Gamma[L := 1] \vdash C$ then $\Gamma \vdash C$ or $\Gamma \vdash C + L^c$.

Proof.

Idea: we put back $L^c$ in the clauses where it was removed.

- If $C \in \Gamma[L := 1]$:
  - either $C$ was in $\Gamma$, thus $\Gamma \vdash C$
  - or $C$ was obtained by removing a $L^c$, thus $\Gamma \vdash C + L^c$

- If $C$ is a resolvent of $A$ and $B$:
Lemma 2.1.22

Let $\Gamma$ a set of clauses, $C$ a clause and $L$ a literal. If $\Gamma[L := 1] \vdash C$ then $\Gamma \vdash C$ or $\Gamma \vdash C + L^c$.

Proof.

Idea: we put back $L^c$ in the clauses where it was removed.

- If $C \in \Gamma[L := 1]$:  
  - either $C$ was in $\Gamma$, thus $\Gamma \vdash C$
  - or $C$ was obtained by removing a $L^c$, thus $\Gamma \vdash C + L^c$

- If $C$ is a resolvent of $A$ and $B$:  
  - either $\Gamma \vdash A$ and $\Gamma \vdash B$, hence $\Gamma \vdash C$
  - or $L^c$ has to be put back into $A$ or $B$, thus into $C$ too

$\square$
Completeness of propositional resolution

Theorem 2.1.24

Let $\Gamma$ be a finite set of clauses. If $\Gamma$ is unsatisfiable then $\Gamma \vdash \bot$.

Proof

By induction on the number of variables in $\Gamma$.

- **Base case:** $\Gamma$ has no variable, so $\Gamma = \emptyset$ (impossible, it's valid) or $\Gamma = \{\bot\}$.

- **Inductive step:** as we know, $\bot$ follows from $\Gamma$ iff it either does from $\Gamma[L := 1]$ or $\Gamma[L := 0]$. Now apply the induction hyp.

  Then we know that either $\Gamma \vdash \bot$, or $\Gamma \vdash x$ and $\Gamma \vdash \overline{x}$.

Corollary 2.1.25

$\Gamma$ is unsatisfiable if and only if $\Gamma \vdash \bot$. 
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Conclusion
Presentation of the two algorithms

How to “systematically” decide whether $\Gamma$ is inconsistent or not?
Presentation of the two algorithms

How to “systematically” decide whether $\Gamma$ is inconsistent or not?

- **Complete strategy**
  
  Purely deductive method: we try to prove $\Gamma \vdash \bot$.
  
  Approach: “intelligent” construction of ALL the deductible clauses (resolvents) from $\Gamma$
Presentation of the two algorithms

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- **The Davis-Putnam-Logemann-Loveland Algorithm**
  Semantic method: we try to build a model for $\Gamma$.
  Approach: “intelligent” traversal of the possible assignments of $\Gamma$
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- **The Davis-Putnam-Logemann-Loveland Algorithm**
  - Semantic method: we try to build a model for $\Gamma$.
  - Approach: “intelligent” traversal of the possible assignments of $\Gamma$

**Remark**

Exponential solutions in time in the worst case.
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Conclusion
Exponential complexity

Remember that two clauses having the same set of literals are equal.

If $\Gamma$ uses $n$, then we have at most $2^n$ distinct clauses deduced from $\Gamma$. 
Reduction of a set of clauses

In order to accelerate the algorithm, we reduce the set of clauses.
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How to proceed with reduction?

Remove the valid clauses and the clauses containing another clause of the set.
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In order to accelerate the algorithm, we \textit{reduce} the set of clauses.

How to proceed with reduction?

Remove the \textit{valid} clauses and the clauses \textit{containing another} clause of the set.

Example 2.1.27

The reduction of the set of clauses \{\(p + q + \bar{p}, p + r, p + r + \bar{s}, r + q\}\} gives the reduced set:
Reduction of a set of clauses

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The reduction of the set of clauses \( \{p + q + \overline{p}, p + r, p + r + \overline{s}, r + q\} \) gives the reduced set:

\[ \{p + q + \overline{p}, p + r, p + r + \overline{s}, r + q\}. \]
Justification

Property 2.1.28

A set of clauses $E$ is equivalent to the reduced set of clauses obtained from $E$. 
Justification

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A set of clauses $E$ is equivalent to the reduced set of clauses obtained from $E$.

Proof.

- Removing valid clauses: $x.1 \equiv x$
Justification

Property 2.1.28

A set of clauses $E$ is equivalent to the reduced set of clauses obtained from $E$.

Proof.

- Removing valid clauses: $x.1 \equiv x$
- Removing a clause including another clause: $x(x + y) \equiv x$
Propositional Resolution
A Deductive Method: Complete Strategy

Principle of the algorithm: Build all the clauses deduced from $\Gamma$

Following the height of the proof trees.

Algorithm

For any integer $i$
While it is possible to construct new clauses
Build the reduced set of all the clauses having a proof tree of height at most $i$. 
Principle of the algorithm: Build all the clauses deduced from $\Gamma$

Following the height of the proof trees.

Algorithm

For any integer $i$
While it is possible to construct new clauses
Build the reduced set of all the clauses having a proof tree of height at most $i$.

In practice:
Maintain two sequences of the sets of clauses, $\Delta_{i(\geq 0)}$ and $\Theta_{i(\geq 0)}$
Result of the algorithm: minimum deduction clauses

Definition 2.1.29

A minimum clause for the deduction from $\Gamma$ is:

- a non-valid clause
- deduced from $\Gamma$
- and containing no other clause deduced from $\Gamma$. 

Example 2.1.30

$\Gamma = \{a + b, b + c + d\}$

The clause $a + c + d$ is a minimum clause for deduction.

But if we add $a + c$ to $\Gamma$, then $a + c + d$ is not minimal anymore (since we can now deduce $c + d$).
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$\Gamma = \{a + b, b + c + d\}$

- The clause $a + c + d$ is a minimum clause for deduction.
- But if we add $\bar{a} + c$ to $\Gamma$, then $a + c + d$ is not minimal anymore (since we can now deduce $c + d$).
Property

Property 2.1.31

Let $\Theta$ be the set of minimum deduction clauses for the set $\Gamma$. $\Gamma$ is unsatisfiable if and only if $\bot \in \Theta$. 
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Proof.

$\Rightarrow$ Suppose $\bot \in \Theta$, then $\Gamma \vdash \bot$, hence by resolution correctness, $\Gamma$ is unsatisfiable.
Property

Property 2.1.31

Let $\Theta$ be the set of minimum deduction clauses for the set $\Gamma$. $\Gamma$ is unsatisfiable if and only if $\bot \in \Theta$.

Proof.

- Suppose $\bot \in \Theta$, then $\Gamma \vdash \bot$, hence by resolution correctness, $\Gamma$ is unsatisfiable.

- Suppose $\Gamma$ is unsatisfiable, by resolution completeness, $\Gamma \vdash \bot$. Consequently $\bot$ is a minimum clause for deduction from $\Gamma$, therefore $\bot \in \Theta$.\[\]
Two sequences of sets of clauses

\( \Delta_i \) are the new useful clauses

- Clauses deduced from \( \Gamma \) by a proof of height \( i \), after removal of:
  - valid clauses
  - clauses including another clause whose proof has height \( < i \).

\( \Delta_0 \) is obtained by reducing \( \Gamma \).
Two sequences of sets of clauses

\( \Delta_i \) are the new useful clauses

Clauses deduced from \( \Gamma \) by a proof of height \( i \), after removal of:

- valid clauses
- clauses including another clause whose proof has height \( < i \).

\( \Delta_0 \) is obtained by reducing \( \Gamma \).

\( \Theta_i \) are the old clauses still useful

Clauses deduced from \( \Gamma \) by a proof of height \( < i \) after removal of:

- valid clauses
- clauses including another clause whose proof has height \( \leq i \).

\( \Theta_0 \) is the empty set.
Construction of the sequences $\Delta_{i(i \geq 0)}$ and $\Theta_{i(i \geq 0)}$

$\Delta_{i+1}$

- Compute all the resolvents of $\Delta_i$ and $\Delta_i \cup \Theta_i$
- Reduce this set
- Remove the new resolvents including a clause from $\Delta_i \cup \Theta_i$

When $\Delta_k = \emptyset$, stop the construction.

$k - 1$ is then the maximum height of a proof.

$\Theta_k$ is the reduced set of the clauses deduced from $\Gamma$. 

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\[ \Theta_{i+1} \]
Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause from $\Delta_{i+1}$. 
Propositional Resolution
A Deductive Method: Complete Strategy

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Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause from $\Delta_{i+1}$.

When $\Delta_k = \emptyset$, stop the construction:

- $k - 1$ is then the maximum height of a proof
- $\Theta_k$ is the reduced set of the clauses deduced from $\Gamma$
Exemple 2.2.1

Soit \( \Gamma = \{ a + b + \overline{a}, a + b, a + b + c, a + \overline{b}, \overline{a} + b, \overline{a} + \overline{b} \} \)

Rappel :

- \( \Delta_{i+1} = \)  
  - Compute all the resolvents of \( \Delta_i \) and \( \Delta_i \cup \Theta_i \)  
  - Reduce this set  
  - Remove the new resolvents which include a clause from \( \Delta_i \cup \Theta_i \)  
- \( \Theta_{i+1} = \)  
  Remove from \( \Delta_i \cup \Theta_i \) the clauses which include a clause of \( \Delta_{i+1} \).
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Soit $\Gamma = \{a + b + \bar{a}, a + b, a + b + c, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b}\}$

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Rappel :

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Rappel :

$\Delta_{i+1} =$

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Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause of $\Delta_{i+1}$. 
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Propositional Resolution

B. Wack et al (UGA)
Exemple 2.2.1

Soit \( \Gamma = \{a + b + \bar{a}, a + b, a + b + c, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b}\} \)

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Soit $\Gamma = \{a+b+\bar{a}, a+b, a+b+c, a+\bar{b}, \bar{a}+b, \bar{a}+\bar{b}\}$

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     |             | $\emptyset$
     | $a + b, a + \bar{b},$
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     |             |                                 | $\bar{a} + b, \bar{a} + \bar{b}$  |
     | $\bar{a} + b, \bar{a} + \bar{b}$        |                                 | $a + \bar{a}, \bar{b}, \bar{a}$  |
| 1   | $a, b, \bar{b}, \bar{a}$                  | $\emptyset$
     | $a, b, \bar{b}, \bar{a}$ | $\perp$ |
| 2   |                                           |             |                        |                                                  |

Rappel :

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Rappel :

$\Delta_{i+1} =$

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Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause of $\Delta_{i+1}$. 
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<th>$\Delta_i \cup \Theta_i$</th>
<th>Résolvants de $\Delta_i$ et $\Delta_i \cup \Theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$a + b + \bar{a}, a + b$</td>
<td>0</td>
<td>$a + b, a + \bar{b}$, $\bar{a} + b, \bar{a} + \bar{b}$</td>
<td>$a, b, b + \bar{b}, a + \bar{a}, \bar{b}, \bar{a}$</td>
</tr>
<tr>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\bar{a} + b, \bar{a} + \bar{b}$</td>
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<tr>
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<td>$a, b, \bar{b}, \bar{a}$</td>
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<td>$a, b, \bar{b}, \bar{a}$</td>
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</tr>
<tr>
<td>2</td>
<td>$\bot$</td>
<td>0</td>
<td>$\bot$</td>
<td>0</td>
</tr>
</tbody>
</table>
Exemple 2.2.1

Soit $\Gamma = \{a + b + \bar{a}, a + b, a + b + c, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b}\}$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\Delta_i$</th>
<th>$\Theta_i$</th>
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</tr>
</thead>
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<tr>
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<td>$a + b + \bar{a}, a + b$</td>
<td>$0$</td>
<td>$a + b, a + b, \bar{a} + b, \bar{a} + \bar{b}$</td>
<td>$a, b, b + b, a + \bar{a}, \bar{b}, \bar{a}$</td>
</tr>
<tr>
<td></td>
<td>$a + b + c, a + \bar{b}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\bar{a} + b, \bar{a} + \bar{b}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$a, b, \bar{b}, \bar{a}$</td>
<td>$0$</td>
<td>$a, b, \bar{b}, \bar{a}$</td>
<td>$\bot$</td>
</tr>
<tr>
<td>2</td>
<td>$\bot$</td>
<td>$0$</td>
<td>$\bot$</td>
<td>$0$</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Rappel :

$\Delta_{i+1} =$

$\Theta_{i+1} =$

Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause of $\Delta_{i+1}$. 

Compute all the resolvents of $\Delta_i$ and $\Delta_i \cup \Theta_i$

Reduce this set

Remove the new resolvents which include a clause from $\Delta_i \cup \Theta_i$
Exemple 2.2.1

Soit \( \Gamma = \{ a + b + \bar{a}, a + b, a + b + c, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b} \} \)

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<tr>
<th>( i )</th>
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<th>Résolvants de ( \Delta_i ) et ( \Delta_i \cup \Theta_i )</th>
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</thead>
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<tr>
<td>0</td>
<td>( a + b + \bar{a}, a + b, a + b + c, a + \bar{b}, \bar{a} + b, \bar{a} + \bar{b} )</td>
<td>0</td>
<td>( a + b, a + b, \bar{a} + \bar{b}, \bar{a} + b, a + \bar{a}, \bar{b}, \bar{a} )</td>
<td></td>
</tr>
<tr>
<td>1</td>
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<td>0</td>
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Rappel :

- \( \Delta_{i+1} = \)
  - Compute all the resolvents of \( \Delta_i \) and \( \Delta_i \cup \Theta_i \)
  - Reduce this set
  - Remove the new resolvents which include a clause from \( \Delta_i \cup \Theta_i \)
- \( \Theta_{i+1} = \)
  - Remove from \( \Delta_i \cup \Theta_i \) the clauses which include a clause of \( \Delta_{i+1} \).
The proof we built

<p>| | |</p>
<table>
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<tr>
<td>1</td>
<td>$a + b$</td>
</tr>
<tr>
<td>2</td>
<td>$a + \overline{b}$</td>
</tr>
<tr>
<td>3</td>
<td>$\overline{a} + b$</td>
</tr>
<tr>
<td>4</td>
<td>$\overline{a} + \overline{b}$</td>
</tr>
<tr>
<td>5</td>
<td>$a$</td>
</tr>
<tr>
<td>6</td>
<td>$b$</td>
</tr>
<tr>
<td>7</td>
<td>$\overline{b}$</td>
</tr>
<tr>
<td>8</td>
<td>$\overline{a}$</td>
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The proof we built

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</table>
Example 2.2.2

\{ a, c, \overline{a} + \overline{b}, \overline{c} + e \}

Rappel :

- \[ \Delta_{i+1} = \]
  - Compute all the resolvents of \( \Delta_i \) and \( \Delta_i \cup \Theta_i \)
  - Reduce this set
  - Remove the new resolvents which include a clause from \( \Delta_i \cup \Theta_i \)

- \[ \Theta_{i+1} = \]
  Remove from \( \Delta_i \cup \Theta_i \) the clauses which include a clause of \( \Delta_{i+1} \).
Example 2.2.2

\[ \{ a, c, \overline{a} + \overline{b}, \overline{c} + e \} \]

<table>
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<tr>
<th>(i)</th>
<th>(\Delta_i)</th>
<th>(\Theta_i)</th>
<th>(\Delta_i \cup \Theta_i)</th>
<th>Rés. de (\Delta_i) et (\Delta_i \cup \Theta_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(a, c, \overline{a} + \overline{b}, \overline{c} + e)</td>
<td>(\emptyset)</td>
<td>(a, c, \overline{a} + \overline{b}, \overline{c} + e)</td>
<td>(b, e)</td>
</tr>
</tbody>
</table>

Rappel :

\[\begin{align*}
\Delta_{i+1} &= \\
&\quad \text{Compute all the resolvents of } \Delta_i \text{ and } \Delta_i \cup \Theta_i \\
&\quad \text{Reduce this set} \\
&\quad \text{Remove the new resolvents which include a clause from } \Delta_i \cup \Theta_i \\
\Theta_{i+1} &= \\
&\quad \text{Remove from } \Delta_i \cup \Theta_i \text{ the clauses which include a clause of } \Delta_{i+1}.
\end{align*}\]
Example 2.2.2

\{ a, c, \overline{a} + \overline{b}, \overline{c} + e \}

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<td>$\emptyset$</td>
<td>$a, c, \overline{a} + \overline{b}, \overline{c} + e$</td>
<td>$b, e$</td>
</tr>
<tr>
<td>1</td>
<td>$\overline{b}, e$</td>
<td>$a, c$</td>
<td>$\overline{b}, e, a, c$</td>
<td>$\emptyset$</td>
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Rappel :

- $\Delta_{i+1} =$
  - Compute all the resolvents of $\Delta_i$ and $\Delta_i \cup \Theta_i$
  - Reduce this set
  - Remove the new resolvents which include a clause from $\Delta_i \cup \Theta_i$

- $\Theta_{i+1} =$
  Remove from $\Delta_i \cup \Theta_i$ the clauses which include a clause of $\Delta_{i+1}$. 
Example 2.2.2

\( \{ a, c, \overline{a} + \overline{b}, \overline{c} + e \} \)

<table>
<thead>
<tr>
<th>( i )</th>
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<td>( \emptyset )</td>
<td>( a, c, \overline{a} + \overline{b}, \overline{c} + e )</td>
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<td>( a, c )</td>
<td>( \overline{b}, e, a, c )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>2</td>
<td>( \emptyset )</td>
<td>( b, e, a, c )</td>
<td></td>
<td></td>
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Rappel :

- \( \Delta_{i+1} = \)
  - Compute all the resolvents of \( \Delta_i \) and \( \Delta_i \cup \Theta_i \)
  - Reduce this set
  - Remove the new resolvents which include a clause from \( \Delta_i \cup \Theta_i \)

- \( \Theta_{i+1} = \)
  - Remove from \( \Delta_i \cup \Theta_i \) the clauses which include a clause of \( \Delta_{i+1} \).
Termination of the algorithm: idea

There are at most $2^n$ clauses deduced from $\Gamma$.

$\Delta_i(i \geq 0)$ contains only clauses deduced from $\Gamma$
Termination of the algorithm: idea

There are at most $2^n$ clauses deduced from $\Gamma$.

$\Delta_i(i \geq 0)$ contains only clauses deduced from $\Gamma$

Property 2.2.4

For all $i \leq k$, the sets $\Delta_i$ are mutually disjoint.
(by construction of $\Delta_i$)
Termination of the algorithm: idea

There are at most $2^n$ clauses deduced from $\Gamma$.

$\Delta_{i(i \geq 0)}$ contains only clauses deduced from $\Gamma$

Property 2.2.4

For all $i \leq k$, the sets $\Delta_i$ are mutually disjoint.
(by construction of $\Delta_i$)

$\Delta_{i(i \geq 0)}$ are mutually disjoint

Hence there are at most $2^n + 1$ sets, therefore $k \leq 2^n + 1$
Result of the algorithm

When the algorithm terminates:

- if $\bot \in \Theta_k : \Gamma$ is unsatisfiable
- if $\bot \notin \Theta_k : \Gamma$ is satisfiable, but what does $\Theta_k$ represent?
Result of the algorithm

When the algorithm terminates:

- if $\bot \in \Theta_k$: $\Gamma$ is unsatisfiable
- if $\bot \notin \Theta_k$: $\Gamma$ is satisfiable, but what does $\Theta_k$ represent?

> $\Theta_k =$ set of minimum deduction clauses.

> $\Gamma$ and $\Theta_k$ are equivalent.
Result of the algorithm

When the algorithm terminates:

if $\bot \in \Theta_k$: $\Gamma$ is unsatisfiable

if $\bot \notin \Theta_k$: $\Gamma$ is satisfiable, but what does $\Theta_k$ represent?

$\Theta_k = \text{set of minimum deduction clauses.}$

$\Gamma$ and $\Theta_k$ are equivalent.

Property 2.2.5

For all $i < k$, the sets $\Delta_i \cup \Theta_i$ and $\Delta_{i+1} \cup \Theta_{i+1}$ are equivalent.

Hence:

$\Gamma \equiv \Delta_0 \cup \emptyset = \Delta_0 \cup \Theta_0 \equiv \ldots \equiv \Delta_k \cup \Theta_k = \emptyset \cup \Theta_k = \Theta_k$
Overview

Correctness of deduction

Completeness of deduction

Introduction to Resolution Algorithms

A Deductive Method: Complete Strategy

A SAT Method: the Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

Conclusion
History

The Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

- Introduced by Martin Davis and Hilary Putnam in 1960, then refined by Martin Davis, George Logemann and Donald Loveland in 1962
- Indicates if a set of clauses is satisfiable and exhibits a model.
- Basis for (most efficient) complete SAT-solvers such as chaff, zchaff and satz.
Principle I

Two types of formulae transformations:

1. preserving the truth value:
   - reduction
## Principle I

### Two types of formulae transformations:

1. **preserving the truth value:**
   - reduction

2. **preserving only satisfiability:**
   - pure literal elimination
   - unit resolution

DPLL is efficient because it uses these two kinds transformations.
Principle II

“Branching/Backtracking” (splitting rule)

- **Branching**: After simplification, assign to **true** a heuristically chosen variable (branching literal).
- **Continue the algorithm recursively.**
Principle II

“Branching/Backtracking” (splitting rule)

- **Branching**: After simplification, assign to **true** a heuristically chosen variable (branching literal).
- Continue the algorithm recursively.
- **Backtracking**: If we arrive to a contradiction, we return to the last choice, and we “branch” by assigning **false** to the chosen variable.
The DPLL Algorithm (figure 2.1)

```python
bool function Algo_DPLL( Γ: set of clauses)
0   Remove the valid clauses from Γ.
    If Γ = ∅, return (true).
    Else return (DPLL(Γ))

bool function DPLL( Γ: set of non-valid clauses)
The function returns true if and only if Γ is satisfiable.
1   If ⊥∈ Γ, return (false).
    If Γ = ∅, return (true).
2   Reduce Γ.
3   Remove from Γ the clauses containing a pure literal.
    If the set Γ has been modified, goto 1.
4   Apply unit resolution to Γ.
    If the set Γ has been modified, goto 1.
5   Pick an arbitrary variable x in Γ
    return (DPLL(Γ[x := 0]) or else DPLL(Γ[x := 1]))
```
The DPLL Algorithm (figure 2.1)

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   return (DPLL(Γ[x := 0]) or else DPLL(Γ[x := 1]))
```
The DPLL Algorithm (figure 2.1)

**bool function** \texttt{Algo\_DPLL}( \Gamma : \text{set of clauses})

0 \hspace{1cm} \text{Remove the valid clauses from } \Gamma.
   \hspace{1cm} \text{If } \Gamma = \emptyset, \text{return (true).}
   \hspace{1cm} \text{Else return (DPLL(}\Gamma))

**bool function** \texttt{DPLL}( \Gamma : \text{set of non-valid clauses})

The function returns true if and only if \Gamma is satisfiable.

1 \hspace{1cm} \text{If } \bot \in \Gamma, \text{return (false).}
   \hspace{1cm} \text{If } \Gamma = \emptyset, \text{return (true).}

2 \hspace{1cm} \text{Reduce } \Gamma.

3 \hspace{1cm} \text{Remove from } \Gamma \text{ the clauses containing a pure literal.}
   \hspace{1cm} \text{If the set } \Gamma \text{ has been modified, goto 1.}

4 \hspace{1cm} \text{Apply unit resolution to } \Gamma.
   \hspace{1cm} \text{If the set } \Gamma \text{ has been modified, goto 1.}

5 \hspace{1cm} \text{Pick an arbitrary variable } x \text{ in } \Gamma
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The DPLL Algorithm (figure 2.1)

**bool function** Algo\_DPLL( $\Gamma$: set of clauses)

0. Remove the valid clauses from $\Gamma$.
   - If $\Gamma = \emptyset$, return (true).
   - Else return (DPLL($\Gamma$))

**bool function** DPLL( $\Gamma$: set of non-valid clauses)

The function returns true if and only if $\Gamma$ is satisfiable.

1. If $\bot \in \Gamma$, return (false).
   - If $\Gamma = \emptyset$, return (true).
2. Reduce $\Gamma$.
3. Remove from $\Gamma$ the clauses containing a pure literal.
   - If the set $\Gamma$ has been modified, goto 1.
4. Apply unit resolution to $\Gamma$.
   - If the set $\Gamma$ has been modified, goto 1.
5. Pick an arbitrary variable $x$ in $\Gamma$
   return (DPLL($\Gamma[x := 0]$) or else DPLL($\Gamma[x := 1]$))
Removal of clauses containing a pure literal

Definition 2.3.1

A literal $L$ is **pure** if none of the clauses in $\Gamma$ contains $L^c$.

Lemma 2.3.2

Removing clauses with a pure literal preserves satisfiability.

Proof: see exercise 49.

Intuition: assigning $[L]_v$ to 1 is always possible for a pure literal.
Example 2.3.3

Let \( \Gamma \) be the set of clauses

1. \( p + q + r \)
2. \( \bar{q} + \bar{r} \)
3. \( q + s \)
4. \( \bar{s} + t \)

Simplify \( \Gamma \) by removing clauses containing pure literals.
Example 2.3.3

Let $\Gamma$ be the set of clauses

1. $p + q + r$
2. $\overline{q} + \overline{r}$
3. $q + s$
4. $\overline{s} + t$

Simplify $\Gamma$ by removing clauses containing pure literals.

The literals $p$ and $t$ are pure.
Therefore we obtain

2. $\overline{q} + \overline{r}$
3. $q + s$
Example 2.3.3

Let $\Gamma$ be the set of clauses

1. $p + q + r$
2. $\overline{q} + \overline{r}$
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4. $\overline{s} + t$

Simplify $\Gamma$ by removing clauses containing pure literals.

The literals $p$ and $t$ are pure. Therefore we obtain

2. $\overline{q} + \overline{r}$

(3) $q + s$

The literals $\overline{r}$ and $s$ are pure.
Example 2.3.3

Let $\Gamma$ be the set of clauses

(1) $p + q + r$
(2) $\overline{q} + \overline{r}$
(3) $q + s$
(4) $\overline{s} + t$

Simplify $\Gamma$ by removing clauses containing pure literals.

The literals $p$ and $t$ are pure.
Therefore we obtain

(2) $\overline{q} + \overline{r}$
(3) $q + s$

The literals $\overline{r}$ and $s$ are pure.
We obtain the empty set.
Example 2.3.3

Let \( \Gamma \) be the set of clauses

\[
\begin{align*}
(1) & \quad p + q + r \\
(2) & \quad \bar{q} + \bar{r} \\
(3) & \quad q + s \\
(4) & \quad \bar{s} + t
\end{align*}
\]

Simplify \( \Gamma \) by removing clauses containing pure literals.

The literals \( p \) and \( t \) are pure. 
Therefore we obtain

\[
\begin{align*}
(2) & \quad \bar{q} + \bar{r} \\
(3) & \quad q + s
\end{align*}
\]

The literals \( \bar{r} \) and \( s \) are pure. 
We obtain the empty set. 
Therefore \( \Gamma \) has a model (for instance \( p = 1, t = 1, r = 0, s = 1 \)).
Unit resolution

Definition 2.3.4

A unit clause is a clause which contains only one literal.
Unit resolution

Definition 2.3.4
A unit clause is a clause which contains only one literal.

Lemma 2.3.5
Let $L$ be the literal from a unit clause of $\Gamma$.
Let $\Theta$ be the set of clauses obtained by:

- removing the clauses containing $L$
- removing $L^c$ inside the remaining clauses
  
  ▶ if $\Gamma$ contains two complementary unit clauses, then $\Theta = \{\bot\}$.

We apply this process for every unit clause.
$\Gamma$ has a model if and only if $\Theta$ has a model.

Proof: The proof is requested in exercise 50.
Example 2.3.6 Unit resolution

Simplify the following sets of clauses by unit resolution:

1. \( \Gamma = p + q, \bar{p}, \bar{q} \)
Example 2.3.6 Unit resolution

Simplify the following sets of clauses by unit resolution:

1. \( \Gamma = p + q, \overline{p}, \overline{q} \)

   - \( q, \overline{q} \) by unit resolution on \( \overline{p} \), then \( \bot \) by UR on \( \overline{q} \)

   Hence \( \Gamma \) has no model.

2. \( \Gamma = a + b + \overline{d}, \overline{a} + c + \overline{d}, \overline{b}, d, \overline{c} \)

   - \( a, a \)
   - \( \bot \) hence \( \Gamma \) has no model.

3. \( \Gamma = p, q, p + r, \overline{p} + r, q + \overline{r}, \overline{q} + s \)

   By unit resolution, we obtain:
   - \( r, s \)

   This set of clauses has a model, hence \( \Gamma \) has a model.
Example 2.3.6 Unit resolution

Simplify the following sets of clauses by unit resolution:

1. \( \Gamma = p + q, \overline{p}, \overline{q} \)
   - \( q, \overline{q} \) by unit resolution on \( \overline{p} \), then \( \bot \) by UR on \( \overline{q} \)
   - Hence \( \Gamma \) has no model.

2. \( \Gamma = a + b + \overline{d}, \overline{a} + c + \overline{d}, \overline{b}, d, \overline{c} \)
   - By unit resolution, we obtain: \( r, s \)
   - This set of clauses has a model, hence \( \Gamma \) has a model.
Example 2.3.6 Unit resolution

Simplify the following sets of clauses by unit resolution:

1. $\Gamma = p + q, \overline{p}, \overline{q}$

   - $q, \overline{q}$ by unit resolution on $\overline{p}$, then $\bot$ by UR on $\overline{q}$
   - Hence $\Gamma$ has no model.

2. $\Gamma = a + b + \overline{d}, \overline{a} + c + \overline{d}, \overline{b}, d, \overline{c}$

   1. $a, \overline{a}$.
Example 2.3.6 Unit resolution

Simplify the following sets of clauses by unit resolution:

1. $\Gamma = p + q, \overline{p}, \overline{q}$
   - $q, \overline{q}$ by unit resolution on $\overline{p}$, then $\bot$ by UR on $\overline{q}$
   - Hence $\Gamma$ has no model.

2. $\Gamma = a + b + \overline{d}, \overline{a} + c + \overline{d}, \overline{b}, d, \overline{c}$
   - 1. $a, \overline{a}$.
   - 2. $\bot$
   - Hence $\Gamma$ has no model.

3. $\Gamma = p, q, p + r, \overline{p} + r, q + \overline{r}, \overline{q} + s$
Example 2.3.6 Unit resolution

Simplify the following sets of clauses by unit resolution:

1. $\Gamma = p + q, \bar{p}, \bar{q}$
   - $q, \bar{q}$ by unit resolution on $\bar{p}$, then $\bot$ by UR on $\bar{q}$
   - Hence $\Gamma$ has no model.

2. $\Gamma = a + b + \bar{d}, \bar{a} + c + \bar{d}, \bar{b}, d, \bar{c}$
   - 1. $a, \bar{a}$.
   - 2. $\bot$
   - Hence $\Gamma$ has no model.

3. $\Gamma = p, q, p + r, \bar{p} + r, q + \bar{r}, \bar{q} + s$
   - By unit resolution, we obtain: $r, s$. 
Example 2.3.6 Unit resolution

Simplify the following sets of clauses by unit resolution:

1. \( \Gamma = p + q, \overline{p}, \overline{q} \)
   - \( q, \overline{q} \) by unit resolution on \( \overline{p} \), then \( \bot \) by UR on \( \overline{q} \)
   - Hence \( \Gamma \) has no model.

2. \( \Gamma = a + b + \overline{d}, \overline{a} + c + \overline{d}, \overline{b}, d, \overline{c} \)
   - 1. \( a, \overline{a} \).
   - 2. \( \bot \)
   - hence \( \Gamma \) has no model.

3. \( \Gamma = p, q, p + r, \overline{p} + r, q + \overline{r}, \overline{q} + s \)
   - By unit resolution, we obtain: \( r, s \).
   - This set of clauses has a model, hence \( \Gamma \) has a model.
Removal of valid clauses

Lemma 2.3.7

Let $\Theta$ be the set of clauses obtained by removing the valid clauses of $\Gamma$.

$\Gamma$ has a model iff $\Theta$ has a model.

Proof.

$\Rightarrow$ Every model of $\Gamma$ is clearly a model of $\Theta$, since $\Theta \subseteq \Gamma$.
Removal of valid clauses

Lemma 2.3.7

Let $\Theta$ be the set of clauses obtained by removing the valid clauses of $\Gamma$.

\[ \Gamma \text{ has a model iff } \Theta \text{ has a model.} \]

Proof.

$\Rightarrow$ Every model of $\Gamma$ is clearly a model of $\Theta$, since $\Theta \subseteq \Gamma$.

$\Leftarrow$ Suppose that $\Theta$ has a model $\nu$.

Let $\nu'$ be the truth assignment built from $\nu$ by assigning any value to the variables appearing in $\Gamma$ but not in $\Theta$.

Every clause $C$ in $\Gamma$ is:

- either a clause of $\Theta$, then $[C]_{\nu'} = [C]_{\nu} = 1$
- or a valid clause, so obviously $\nu'$ is a model of $C$.

Hence $\nu'$ is a model of $\Gamma$. 
The DPLL Algorithm (figure 2.1)

bool function Algo_DPLL( \( \Gamma \): set of clauses)
0 Remove the valid clauses from \( \Gamma \).
   If \( \Gamma = \emptyset \), return (true).
   Else return (DPLL(\( \Gamma \)))

bool function DPLL( \( \Gamma \): set of non-valid clauses)
The function returns true if and only if \( \Gamma \) is satisfiable.
1 If \( \bot \in \Gamma \), return(false).
   If \( \Gamma = \emptyset \), return (true).
2 Reduce \( \Gamma \).
3 Remove from \( \Gamma \) the clauses containing a pure literal.
   If the set \( \Gamma \) has been modified, goto 1.
4 Apply unit resolution to \( \Gamma \).
   If the set \( \Gamma \) has been modified, goto 1.
5 Pick an arbitrary variable \( x \) in \( \Gamma \)
return (DPLL(\( \Gamma \mid x := 0 \)) or else DPLL(\( \Gamma \mid x := 1 \)))
Example 2.3.8

Let $\Gamma$ be the set of clauses: $\overline{a} + \overline{b}$, $a + b$, $\overline{a} + \overline{c}$, $a + c$, $\overline{b} + \overline{c}$, $b + c$. Since every leaf contains the empty clause, the set $\Gamma$ is unsatisfiable.
Example 2.3.8

Let $\Gamma$ be the set of clauses: $\overline{a} + \overline{b}$, $a + b$, $\overline{a} + \overline{c}$, $a + c$, $\overline{b} + \overline{c}$, $b + c$. 

\[
\begin{align*}
\overline{a} + \overline{b}, & \quad a + b, \quad \overline{a} + \overline{c}, \quad a + c, \quad \overline{b} + \overline{c}, \quad b + c
\end{align*}
\]
Example 2.3.8

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Let $\Gamma$ be the set of clauses: $\overline{a} + b$, $a + b$, $\overline{a} + \overline{c}$, $a + c$, $\overline{b} + \overline{c}$, $b + c$. 

\begin{itemize}
  \item $a = 0$ \Rightarrow $\overline{a} + b, a + b, \overline{a} + \overline{c}, a + c, \overline{b} + \overline{c}, b + c$
  \item $a = 1$ \Rightarrow $\overline{b}, \overline{c}, \overline{b} + \overline{c}, b + c$
\end{itemize}

Since every leave contains the empty clause, the set $\Gamma$ is unsatisfiable.
Example 2.3.8

Let $\Gamma$ be the set of clauses: $\bar{a} + \bar{b}$, $a + b$, $\bar{a} + \bar{c}$, $a + c$, $\bar{b} + \bar{c}$, $b + c$.

Since every leaf contains the empty clause, the set $\Gamma$ is unsatisfiable.
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Since every leaf contains the empty clause, the set $\Gamma$ is unsatisfiable.
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Let \( \Gamma \) be the set of clauses: \( \overline{a} + \overline{b}, a + b, \overline{a} + \overline{c}, a + c, \overline{b} + \overline{c}, b + c \).

Since every leave contains the empty clause, the set \( \Gamma \) is unsatisfiable.
Example 2.3.8

Let $\Gamma$ be the set of clauses: $\overline{p} + q$, $\overline{p} + s$, $p + q$, $\overline{p} + \overline{s}$. 
Example 2.3.8

Let \( \Gamma \) be the set of clauses: \( \overline{p} + q, \overline{p} + s, p + q, \overline{p} + \overline{s} \).

Since one branch leads to the empty set, the set \( \Gamma \) is satisfiable. It is **useless** to continue the construction of the right branch.
Theorems 2.3.9 et 2.3.10

The algorithm Algo\_DPLL is correct and terminates.
The algorithm Algo\_DPLL is correct and terminates.

**Termination proof**

- Valid clause removal is only executed once
- Simplification iteration: the number of clauses strictly decreases
- Recursive calls: the number of variables strictly decreases

Hence the termination.
Correctness proof

- Invariant for the simplification loop:
  the current value of $\Gamma$ has a model iff $\Gamma$ has a model.
Invariant for the simplification loop:

the current value of $\Gamma$ has a model iff $\Gamma$ has a model.

see lemma for each simplification.
Correctness proof

- Invariant for the simplification loop:
  the current value of \( \Gamma \) has a model iff \( \Gamma \) has a model.
  see lemma for each simplification.

- Correctness of recursive calls:
  Reminder of property 2.1.21:
  \( \Gamma \) has a model iff \( \Gamma[x := 0] \) or \( \Gamma[x := 1] \) is satisfiable.
  So if the recursive calls are correct, the current call is too.
Correctness proof

- Invariant for the simplification loop:
  the current value of $\Gamma$ has a model iff $\Gamma$ has a model.
  see lemma for each simplification.

- Correctness of recursive calls:

  Reminder of property 2.1.21:
  $\Gamma$ has a model iff $\Gamma[x := 0]$ or $\Gamma[x := 1]$ is satisfiable.
  So if the recursive calls are correct, the current call is too.

  Since the algorithm is correct for a set $\Gamma$ with no literal, it is correct for any set $\Gamma$ of clauses.
Remarks 2.3.11 and 2.3.12

▶ **Forgetting simplifications:** DPLL is still correct if we forget (once or more) reduction (2), pure literal elimination (3) and/or unit reduction (4).
Remarks 2.3.11 and 2.3.12

- **Forgetting simplifications:** DPLL is still correct if we forget (once or more) reduction (2), pure literal elimination (3) and/or unit reduction (4).

- **Choice of the variable (branching literal):**
  - A good choice for variable $x$ in step (5) is the variable that appears most often.
  - A better choice is the variable which will lead to the maximum number of simplifications

Cf. Sub-section 2.3.5, for the main branching heuristics
Overview

Correctness of deduction

Completeness of deduction

Introduction to Resolution Algorithms

A Deductive Method: Complete Strategy

A SAT Method: the Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

Conclusion
Today

- Resolution is a **correct and complete** deductive system: it **characterizes** all the unsatisfiable formulae.
- **Complete Strategy** is an **algorithm** for computing **every** clause deducible from an initial set.
- The **DPLL algorithm** uses ideas from resolution to:
  - find a **model**
  - or else, prove the **unsatisfiability** by an efficient search of the assignments.
Next lecture

- Natural deduction

Homework: **Hypotheses**: 

- (H1) : \( p \Rightarrow \neg j \equiv \neg p \lor \neg j \)
- (H2) : \( \neg p \Rightarrow j \equiv p \lor j \)
- (H3) : \( j \Rightarrow m \equiv \neg j \lor m \)
- (¬ C): \( \neg m \land \neg p \) (two clauses)

**Build** the proof of \( H1, H2, H3, \neg C \vdash \bot \) obtained by the DPLL algorithm (you may pick any variable for branching)