

Natural Deduction

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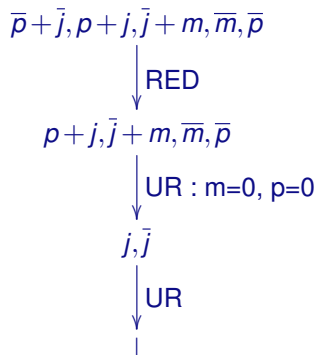
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Last course

- ▶ Correctness and completeness of resolution
- ▶ Complete Strategy
- ▶ Davis-Putnam algorithm

Homework: solution with DPLL



Plan

Introduction to natural deduction

Rules

Natural deduction proofs

Conclusion

Intuition

When we write proofs in math courses,
when we decompose a reasoning in elementary obvious steps,
we practice **natural deduction**.

Natural Deduction (ND)

New deductive systems introduced by Gentzen (1934) :

▶ **Natural deduction:**

- ▶ we prove consequences $\Gamma \vdash p$ rather than tautologies
- ▶ only one axiom $\Gamma, p \vdash p$
- ▶ introduction and elimination rules for each connective



▶ **Sequent calculus:**

- ▶ $\Gamma \vdash \Delta$ if whenever all of Γ is true, one of the formulas in Δ is true
- ▶ left and right introduction rules
- ▶ *cut* rule
$$\frac{\Gamma \vdash \Delta, p \quad \Gamma', p \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

▶ Computing with proofs: **cut elimination**

Every proof that does not use the excluded middle can be transformed into a **constructive** proof.

Resolution vs. Natural deduction

A proof by **resolution** is a list of clauses built using **any of the previous clauses**.

In **natural deduction**, during a proof, **we can add and remove hypotheses**.

Abbreviations

\top , negation and equivalence are **abbreviations** defined as:

- ▶ \top abbreviates $\perp \Rightarrow \perp$.
- ▶ $\neg A$ abbreviates $A \Rightarrow \perp$.
- ▶ $A \Leftrightarrow B$ abbreviates $(A \Rightarrow B) \wedge (B \Rightarrow A)$.

Two formulae are considered to be **equal**, if the formulas obtained by removing the abbreviations are identical.

For example, the formulae $\neg\neg a$, $\neg a \Rightarrow \perp$ and $(a \Rightarrow \perp) \Rightarrow \perp$ are equal.

Two equal formulae are equivalent!

Rule

Definition 3.1.1

A **rule** consists of:

- ▶ some formulae H_1, \dots, H_n called **premises** (or hypotheses)
- ▶ a unique **conclusion** C
- ▶ sometimes a name R for the rule

$$\frac{H_1 \dots H_n}{C} R$$

Example : Proof of a conjunction

$$\frac{A \quad B}{A \wedge B} (\wedge I)$$

Classification of rules

- ▶ **Introduction rules** for introducing a connective in the conclusion.
- ▶ **Elimination rules** for removing a connective from one of the premises.
- ▶ + **two special rules**

The rules (system NK of Gentzen)

Table 3.1

	Introduction	Elimination
Implication	$\begin{array}{c} [A] \\ \vdots \\ B \\ \hline A \Rightarrow B \end{array} \quad \Rightarrow I$	$\frac{A \quad A \Rightarrow B}{B} \quad \Rightarrow E$
Conjunction	$\frac{A \quad B}{A \wedge B} \quad \wedge I$	$\frac{A \wedge B}{A} \quad \wedge E1$ $\frac{A \wedge B}{B} \quad \wedge E2$
Disjunction	$\frac{A}{A \vee B} \quad \vee I1$ $\frac{B}{A \vee B} \quad \vee I2$	$\frac{A \vee B \quad A \Rightarrow C \quad B \Rightarrow C}{C} \quad \vee E$
\perp	Ex falso quodlibet	
	$\frac{\perp}{A} \quad Efq$	
	Reductio ad absurdum	
	$\frac{\neg \neg A}{A} \quad RAA$	

[A] means that A is a hypothesis

A “simple” example

$$\frac{\frac{A \quad A \Rightarrow B}{B} \Rightarrow E \quad \frac{A \quad A \Rightarrow C}{C} \Rightarrow E}{B \wedge C} \wedge I$$

What have we proven here exactly? $B \wedge C$

under the hypotheses $A, A \Rightarrow B, A \Rightarrow C$

i.e. $A, A \Rightarrow B, A \Rightarrow C \vdash B \wedge C$

Fundamental rule of Natural Deduction

Implies-introduction:

In order to prove $A \Rightarrow B$,

just derive B with the additional hypothesis A and then remove this assumption.

(If $A \models B$ then $\models A \Rightarrow B$)

$$\begin{array}{c}
 [A] \quad H_1 \quad \dots \quad H_n \\
 \quad \quad \vdots \\
 \quad \quad B \\
 \hline
 A \Rightarrow B \quad \Rightarrow I \\
 H_1, \dots, H_n \models A \Rightarrow B.
 \end{array}
 \quad \text{proves that}$$

Proof line

Definition 3.1.2

A proof **line** is one of the three following:

- ▶ Assume **formula** (to add an hypothesis)
- ▶ **formula** (derived from previous lines using the rules)
- ▶ Therefore **formula** (to remove the last hypothesis)

This last case is **the rule of implies-introduction**.

Examples:

▶ Assume $A \wedge B$

▶ A

▶ Therefore $A \wedge B \Rightarrow A$

$$\frac{\frac{[A \wedge B]}{A} \wedge E}{A \wedge B \Rightarrow A} \Rightarrow I$$

Proof sketch

Definition 3.1.3

A proof **sketch** is a sequence of lines such that, in every prefix of the sequence, there are at least as many Assume as Therefore.

Example 3.1.4

number	line
1	Assume a
2	$a \vee b$
3	Therefore $a \Rightarrow a \vee b$
4	Therefore $\neg a$
5	Assume b

Proof sketch: examples

Where are the sketches?

num	line
1	Assume $a \wedge b$
2	b
3	$b \vee c$
4	Therefore $a \wedge b \Rightarrow b \vee c$
5	Therefore $\neg a$
6	Assume b

num	line
1	Assume a
2	$a \vee b$
3	Therefore $a \Rightarrow a \vee b$
4	Assume b
5	Therefore $\neg a$

num	line
1	Assume a
2	$a \vee b$
3	Therefore $a \Rightarrow a \vee b$
4	Assume b

Context (1/2)

- ▶ Each line of a proof sketch has a **context**
- ▶ The **context** is the sequence of hypotheses previously introduced in *Assume* lines and not removed in *Therefore* lines.

Example 3.1.6:

context	number	line	rule
1	1	Assume a	
1,2	2	Assume b	
1,2	3	$a \wedge b$	$\wedge I$ 1,2
1	4	Therefore $b \Rightarrow a \wedge b$	$\Rightarrow I$ 2,3
1,5	5	Assume e	

Context (2/2)

The context of a formula represents the hypotheses from which it has been derived.

Definition 3.1.5

Formally: Γ_i is the context of the line i .

$$\Gamma_0 = \emptyset$$

If the line i is:

- ▶ Assume A
then $\Gamma_i = \Gamma_{i-1}, i$
- ▶ A
then $\Gamma_i = \Gamma_{i-1}$
- ▶ Therefore A
then Γ_i is obtained by deleting the **last** formula in Γ_{i-1}

Example of context

Write down the **contexts** of the following proof sketch:

context	number	line
1	1	Assume a
1	2	$a \vee b$
	3	Therefore $a \Rightarrow a \vee b$
4	4	Assume b
	5	Therefore b

Usable formulae (1/2)

Definition 3.1.7

- ▶ A formula appearing on a line of a proof sketch is its **conclusion**.
- ▶ The conclusion of a line is **usable** as long as its context (*i.e.*, the hypotheses from which it has been derived) is present.

Example 3.1.8

context	number	line
1	1	Assume a
1	2	$a \vee b$
	3	Therefore $a \Rightarrow b$
	4	a
	5	$b \vee a$

The conclusion of line 2 is usable on line 2 and not beyond.

Usable formulae (2/2)

On which lines are formulae 1 and 3 **usable**?

context	number	line
1	1	Assume a
1,2	2	Assume b
1,2	3	c
1	4	Therefore d
1,5	5	Assume e

Definition of a Proof

Definition 3.1.9

Let Γ be a set of formulae.

A **proof in the environment** Γ is a proof sketch such that:

1. For every “Therefore” line, the formula is $B \Rightarrow C$, where:
 - ▶ B is the last hypothesis we’ve removed (from the context of the previous line)
 - ▶ C is either a formula usable on the previous line, or belongs to Γ .
2. For every “A” line, the formula A is:
 - ▶ the conclusion of a rule (other than $\Rightarrow I$)
 - ▶ whose premises are usable on the previous line, or belong to Γ .

Beware:

- ▶ The **context** Γ_i changes during the proof.
- ▶ The **environment** Γ remains the same.

Proof of formulae

Definition 3.1.10

A **proof of formula** A within the environment Γ is:

- ▶ either the empty proof (when A is an element of Γ),
- ▶ or a proof whose last line is A with an empty context.

We note:

- ▶ $\Gamma \vdash A$ the fact that there is a proof of A within the environment Γ ,
- ▶ $\Gamma \vdash P : A$ the fact that P is a proof of A within Γ .
- ▶ When the environment is empty, we abbreviate $\emptyset \vdash A$ by $\vdash A$.
- ▶ When we ask for a proof without indicating the environment, we mean that $\Gamma = \emptyset$.

First Example (exemple 3.1.11)

Let us prove $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$.

context	number	proof	justification
context	number	proof	justification
1	1	Assume $a \Rightarrow b$	
context	number	proof	justification
1	1	Assume $a \Rightarrow b$	
1,2	2	Assume $\neg b$	
context	number	proof	justification
1	1	Assume $a \Rightarrow b$	
1,2	2	Assume $\neg b$	
1,2,3	3	Assume a	
context	number	proof	justification
1	1	Assume $a \Rightarrow b$	
1,2	2	Assume $\neg b$	

Proofs with abbreviations vs. without abbreviations

cont.	n.	proof with abbreviation	proof without abbreviation
1	1	Assume $a \Rightarrow b$	Assume $a \Rightarrow b$
1,2	2	Assume $\neg b$	Assume $b \Rightarrow \perp$
1,2,3	3	Assume a	Assume a
1,2,3	4	b	b
1,2,3	5	\perp	\perp
1,2	6	Therefore $\neg a$	Therefore $a \Rightarrow \perp$
1	7	Therefore $\neg b \Rightarrow \neg a$	Therefore $(b \Rightarrow \perp) \Rightarrow (a \Rightarrow \perp)$
	8	Therefore $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$	Therefore $(a \Rightarrow b) \Rightarrow ((b \Rightarrow \perp) \Rightarrow (a \Rightarrow \perp))$

Second Example

Prove $a \wedge \neg a \Rightarrow b$.

context	number	proof	justification
context	number	proof	justification
1	1	Assume $a \wedge \neg a$	
context	number	proof	justification
1	1	Assume $a \wedge \neg a$	
1	2	a	$\wedge E1$ 1
context	number	proof	justification
1	1	Assume $a \wedge \neg a$	
1	2	a	$\wedge E1$ 1
1	3	$\neg a$	$\wedge E2$ 1
context	number	proof	justification
1	1	Assume $a \wedge \neg a$	
1	2		$\wedge E1$ 1

Proofs with abbreviations vs. without abbreviation (2/2)

contexte	number	proof with abbreviation	proof without abbreviation	justification
1	1	Assume $a \wedge \neg a$	Assume $a \wedge (a \Rightarrow \perp)$	
1	2	a	a	$\wedge E1$ 1
1	3	$\neg a$	$a \Rightarrow \perp$	$\wedge E2$ 1
1	4	\perp	\perp	$\Rightarrow E$ 2,3
1	5	b	b	$Efq4$
	6	Therefore $a \wedge \neg a \Rightarrow b$	Therefore $a \wedge (a \Rightarrow \perp) \Rightarrow b$	$\Rightarrow I$ 1,5

Today

- ▶ **Propositional natural deduction** reflects the usual **deduction rules** into a formal system.
- ▶ Unlike in resolution, a proof occurs in a **context** (list of formulae **assumed** at a given point).

Next lecture

- ▶ Completeness
- ▶ Correctness
- ▶ Tactics

Homework: prove

$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

using natural deduction.