Natural Deduction

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Last course

- Correctness and completeness of resolution
- Davis-Putnam algorithm

Homework: solution with DPLL

$$\begin{array}{c}
\overline{p}+\bar{j}, p+j, \bar{j}+m, \overline{m}, \overline{p} \\
\Big\downarrow \text{RED} \\
p+j, \bar{j}+m, \overline{m}, \overline{p} \\
\Big\downarrow \text{UR : m=0, p=0} \\
j, \bar{j} \\
\Big\downarrow \text{UR} \\
\perp
\end{array}$$

Plan

Intuition

Resolution is a way of writing formal proofs which can be:

- verified
- or even generated

automatically

... but the proofs are hard to read.

When we write a proof “by hand”,
each elementary step is easy to understand.

Natural deduction aims at producing **formal proofs** whose elementary steps are **easy to read**.

Natural Deduction (ND)

New deductive systems (1934) introduced by Gentzen (1909-45):

- **Natural deduction**:
  - we prove consequences $\Gamma \vdash p$ rather than tautologies
  - only one axiom $\Gamma, \ p \vdash p$
  - introduction and elimination rules for each connective

- **Sequent calculus**:
  - $\Gamma \vdash \Delta$ if whenever all of Γ is true, one of the formulas in Δ is true
  - left and right introduction rules
  - *cut* rule $\dfrac{\Gamma \vdash \Delta, p \quad \Gamma', p \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$
- Computing with proofs: **cut elimination**
  Every proof that does not use the excluded middle can be transformed into a **constructive** proof.

Rule

Definition 3.1.1

A rule consists of:

- some formulas $H_1, \ldots, H_n$ called **premises** (or hypotheses)
- a unique **conclusion** C
- sometimes a name R for the rule

$$\frac{H_1 \ldots H_n}{C}\ R$$

Example : Proof of a conjunction

$$\frac{A \quad B}{A \wedge B}\ (\wedge I)$$

Classification of rules

- **Introduction rules** for introducing a connective in the conclusion.
- **Elimination rules** for removing a connective from one of the premises.
- $+$ **two special rules**

The rules (system NK of Gentzen)

Table 3.1

	Introduction	Elimination
Implication	$\dfrac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \Rightarrow B} \Rightarrow I$	$\dfrac{A \quad A \Rightarrow B}{B} \Rightarrow E$
Conjunction	$\dfrac{A \quad B}{A \wedge B} \wedge I$	$\dfrac{A \wedge B}{A} \wedge E1$ $\dfrac{A \wedge B}{B} \wedge E2$
Disjunction	$\dfrac{A}{A \vee B} \vee I1$ $\dfrac{B}{A \vee B} \vee I2$	$\dfrac{A \vee B \quad A \Rightarrow C \quad B \Rightarrow C}{C} \vee E$
	Ex falso quodlibet	
$\perp$	$\dfrac{\perp}{A}$ *Efq*	
	Reductio ad absurdum	
	$\dfrac{\neg\neg A}{A}$ *RAA*	

Abbreviations

$\top$, negation and equivalence are abbreviations defined as:

- $\top$ abbreviates $\bot \Rightarrow \bot$.
- $\neg A$ abbreviates $A \Rightarrow \bot$.
- $A \Leftrightarrow B$ abbreviates $(A \Rightarrow B) \wedge (B \Rightarrow A)$.

Two formulas are considered to be equal, if the formulas obtained by removing the abbreviations are identical.

For example, the formulas $\neg\neg a$, $\neg a \Rightarrow \bot$ and $(a \Rightarrow \bot) \Rightarrow \bot$ are equal.

Two equal formulas are equivalent!

A “simple” example

$$\dfrac{\dfrac{A \qquad A \Rightarrow B}{B} \Rightarrow E \qquad \dfrac{A \qquad A \Rightarrow C}{C} \Rightarrow E}{B \wedge C} \wedge I$$

What have we proven here exactly? $B \wedge C$
under the hypotheses $A,\ A \Rightarrow B,\ A \Rightarrow C$

i.e. $A,\ A \Rightarrow B,\ A \Rightarrow C \models B \wedge C$

Attempt at defining a proof

Let Γ be a set of formulas : the hypotheses.

A proof in the environment Γ is a sequence of formulas such that, on every line, the formula A is:

- the conclusion of a rule in the system
- whose premises are:
  - available in Γ
  - or available on a previous line.

An example with an environment

Let us prove $C \wedge D$ in the environment $\{A \wedge B,\ \ A \Rightarrow C,\ \ B \Rightarrow D\}$

environment		
reference	formula	
(i)	$A \wedge B$	
(ii)	$A \Rightarrow C$	
(iii)	$B \Rightarrow D$	
number	proof	justification
1	A	$\wedge E\ (i)$
2	C	$\Rightarrow E\ 1, (ii)$
3	B	$\wedge E\ (i)$
4	D	$\Rightarrow E\ 3, (iii)$
5	$C \wedge D$	$\wedge I\ 2, 4$

How to prove an implication?

Build a proof of $A \Rightarrow (B \Rightarrow A)$ without environment.

Principle : use A to prove $B \vee A$.
In a proof, there will be lines looking like `Assume` formula.

number	proof	justification
1	`Assume` A	
2	$B \vee A$	$\vee I$ 1
3	$A \Rightarrow B \vee A$	$\Rightarrow I$ 1,2

The fundamental rule of Natural Deduction

Implies-introduction:

In order to prove $A \Rightarrow B$,
just **derive** B **with the additional hypothesis** A
and then remove this assumption.

$$(\text{If } A \models B \text{ then } \models A \Rightarrow B)$$

$$\dfrac{\begin{matrix}[A] & & H_1 & \dots & H_n \\ & \ddots & \vdots & \iddots & \\ & & B & & \end{matrix}}{A \Rightarrow B} \Rightarrow I$$

proves that $H_1, \dots, H_n \vDash A \Rightarrow B$.

Building tree-like proofs

To practice building tree-like proofs, one can use:

http://www-sop.inria.fr/marelle/Laurent.Thery/peanoware/Nd.html

Notion of context (1/2)

- The context is the sequence of hypotheses previously introduced and not yet integrated in an implication
- One context for each proof line
- To abbreviate, we mark numbers rather than formulas

Example 3.1.6:

context	number	line	rule
1	1	Assume a	
1,2	2	Assume b	
1,2	3	$a \wedge b$	$\wedge$I 1,2
1	4	Therefore $b \Rightarrow a \wedge b$	$\Rightarrow$I 2,3
1,5	5	Assume e	

Proof line

Definition 3.1.2

A proof line is one of the three following:

- `Assume` formula (to add an hypothesis)
- formula (obtained using the rules)
- `Therefore` formula $\Rightarrow$ formula (which removes the last hypothesis)

This last case is the rule of implies-introduction.

Examples:

- `Assume` $A \wedge B$
- A
- `Therefore` $A \wedge B \Rightarrow A$

$$\dfrac{\dfrac{[A \wedge B]}{A}\,\wedge E}{A \wedge B \Rightarrow A}\Rightarrow I$$

Consequence over the structure of proofs

Definition 3.1.3

At each step of a proof,
there must be at least as many `Assume` as `Therefore`.

Example 3.1.4

context	number	line
1	1	`Assume` a
1	2	$a \vee b$
	3	`Therefore` $a \Rightarrow a \vee b$
??	4	`Therefore` $\neg a$
	5	`Assume` b

Context (2/2)

The context of a formula represents the hypotheses from which it has been derived.

Definition 3.1.5

Initially the context is empty.

If the line i is:

- `Assume` A
  then we add formula i to the previous context
- A
  then we keep the previous context
- `Therefore` $A \Rightarrow B$
  then we delete the last formula in the previous context

Example of context

Write down the **contexts** of the following proof sketch:

context	number	line
1	1	`Assume` a
1	2	$a \vee b$
	3	`Therefore` $a \Rightarrow a \vee b$
4	4	`Assume` b
	5	`Therefore` b

Caution : hypotheses may end up polluting the proof.
(What prevents us from deducing $a \Rightarrow b$ here?)

Resolution *vs.* Natural deduction

In a proof by **resolution**, each clause is built using any of the previous clauses.

In **natural deduction**, during a proof, we can add and remove hypotheses. Thus formulas can **not** always be used freely.

Usable formulas (1/2)

Definition 3.1.7

- The formula appearing on a line is usable as long as its context (*i.e.*, the hypotheses from which it has been derived) is present.

Example 3.1.8

context	number	line
1	1	`Assume` a
1	2	$a \lor b$
	3	`Therefore` $a \Rightarrow b$
	4	a
	5	$b \lor a$

The formula on line 2 is usable on line 2 and not beyond.

Usable formulas (2/2)

On which lines are formulas 1 and 3 **usable**?

context	number	line
1	1	`Assume` a
1,2	2	`Assume` b
1,2	3	c
1	4	`Therefore` d
1,5	5	`Assume` e

Definition of a Proof in Natural Deduction

Definition 3.1.9

A proof in natural deduction is a sequence of proof lines such that:

1. For every "A" line, the formula is:
   - the conclusion of a rule in the system (other than $\Rightarrow I$)
   - whose premises are usable on the previous line
2. For every "`Therefore` $B \Rightarrow C$" line:
   - B is the hypothesis removed from the context (its last formula)
   - C is a formula usable on the previous line

The last line in the proof must have an empty context ;
its formula is the conclusion of the proof.

Moreover, we may have an environment Γ of formulas (hypotheses) that are usable on any line.
(not to be confused with the context, which changes during the proof)

Example 3.1.11

Let us prove $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$.

contexte	numéro	preuve	justification
1	1	Assume $a \Rightarrow b$	
1,2	2	Assume $\neg b$	
1,2,3	3	Assume a	
1,2,3	4	b	$\Rightarrow E$ 1, 3
1,2,3	5	$\perp$	$\Rightarrow E$ 2, 4
1,2	6	Therefore $\neg a$	$\Rightarrow I$ 3, 5
1	7	Therefore $\neg b \Rightarrow \neg a$	$\Rightarrow I$ 2, 6
	8	Therefore $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$	$\Rightarrow I$ 1,7

Proofs with abbreviations *vs.* without abbreviations

cont.	n.	proof with abbreviation	proof without abbreviation
1	1	`Assume` $a \Rightarrow b$	`Assume` $a \Rightarrow b$
1,2	2	`Assume` $\neg b$	`Assume` $b \Rightarrow \perp$
1,2,3	3	`Assume` a	`Assume` a
1,2,3	4	b	b
1,2,3	5	$\perp$	$\perp$
1,2	6	`Therefore` $\neg a$	`Therefore` $a \Rightarrow \perp$
1	7	`Therefore` $\neg b \Rightarrow \neg a$	`Therefore` $(b \Rightarrow \perp) \Rightarrow (a \Rightarrow \perp)$
	8	`Therefore` $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$	`Therefore` $(a \Rightarrow b) \Rightarrow ((b \Rightarrow \perp) \Rightarrow (a \Rightarrow \perp))$

Tree (example 3.1.11)

$$
\dfrac{\dfrac{\dfrac{\dfrac{(2)\cancel{\neg b} \qquad \dfrac{(1)\cancel{a \Rightarrow b} \qquad (3)\cancel{a}}{(4) b} \Rightarrow E}{(5)\perp} \Rightarrow E}{(6)\neg a} \Rightarrow I[3]}{(7)\neg b \Rightarrow \neg a} \Rightarrow I[2]}{(8)(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)} \Rightarrow I[1]
$$

context	number	proof	justification
1	1	Assume $a \Rightarrow b$	
1,2	2	Assume $\neg b$	
1,2,3	3	Assume a	
1,2,3	4	b	$\Rightarrow E$ 1, 3
1,2,3	5	$\perp$	$\Rightarrow E$ 2, 4
1,2	6	Therefore $\neg a$	$\Rightarrow I$ 3, 5
1	7	Therefore $\neg b \Rightarrow \neg a$	$\Rightarrow I$ 2, 6
	8	Therefore $(a \Rightarrow b) \Rightarrow (\neg b \Rightarrow \neg a)$	$\Rightarrow I$ 1,7

Second Example

Prove $a \wedge \neg a \Rightarrow b$.

context	number	proof	justification
1	1	`Assume` $a \wedge \neg a$	
1	2	a	$\wedge E1$ 1
1	3	$\neg a$	$\wedge E2$ 1
1	4	$\perp$	$\Rightarrow E$ $2, 3$
1	5	b	Efq 4
	6	`Therefore` $a \wedge \neg a \Rightarrow b$	$\Rightarrow I$ $1, 5$

Proofs with abbreviations *vs.* without abbreviation (2/2)

contexte	number	proof with abbreviation	proof without abbreviation	justification
1	1	$\texttt{Assume}\ a \wedge \neg a$	$\texttt{Assume}\ a \wedge (a \Rightarrow \bot)$	
1	2	a	a	$\wedge E1\ 1$
1	3	$\neg a$	$a \Rightarrow \bot$	$\wedge E2\ 1$
1	4	$\bot$	$\bot$	$\Rightarrow E\ 2,3$
1	5	b	b	$Efq4$
	6	$\texttt{Therefore}\ a \wedge \neg a \Rightarrow b$	$\texttt{Therefore}\ a \wedge (a \Rightarrow \bot) \Rightarrow b$	$\Rightarrow I\ 1,5$

Today

- Propositional natural deduction reflects the usual **deduction rules** into a formal system.
- Unlike in resolution, a proof occurs in a context (list of formulas **assumed** at a given point).

Next lecture

- Examples with the other rules
- Completeness
- Correctness
- Tactics

Homework: prove

$$(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m) \Rightarrow m \vee p$$

using natural deduction.