Natural Deduction
Properties and tactics

Benjamin Wack
Paolo Torrini

Université Grenoble Alpes

February 2021
Last lecture

Natural deduction
  ▶ Rules
  ▶ Context
  ▶ Proofs
### Reminder of the rules

<table>
<thead>
<tr>
<th>Introduction</th>
<th>Elimination</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Implication</strong></td>
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</tr>
<tr>
<td>([A])</td>
<td>(A \rightarrow B \quad \Rightarrow E)</td>
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<tr>
<td>(\ldots)</td>
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</tr>
<tr>
<td>(B)</td>
<td></td>
</tr>
<tr>
<td>(\frac{A \rightarrow B}{A \rightarrow B} \quad \Rightarrow I)</td>
<td>(\frac{A \land B}{A \land B} \quad \land I)</td>
</tr>
<tr>
<td>(A)</td>
<td>(A)</td>
</tr>
<tr>
<td>(\frac{A \land B}{A \land B} \quad \land E_1)</td>
<td>(\frac{A \land B}{A \land B} \quad \land E_2)</td>
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<tr>
<td><strong>Conjunction</strong></td>
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<tr>
<td>(A \lor B)</td>
<td>(A \lor B)</td>
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<tr>
<td>(\frac{A \lor B}{A \lor B} \quad \lor I_1)</td>
<td>(A \lor B)</td>
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<tr>
<td>(B)</td>
<td>(A \lor B)</td>
</tr>
<tr>
<td>(\frac{A \lor B}{A \lor B} \quad \lor I_2)</td>
<td>(A \lor B)</td>
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<tr>
<td><strong>Ex falso quodlibet</strong></td>
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</tr>
<tr>
<td>(\bot)</td>
<td>(\frac{\bot}{A} \quad \text{Efq})</td>
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<tr>
<td><strong>Disjunction</strong></td>
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<tr>
<td>(A)</td>
<td>(A)</td>
</tr>
<tr>
<td>(\frac{A \lor B}{A \lor B} \quad \lor E)</td>
<td>(\frac{A \lor B}{A \lor B} \quad \lor E)</td>
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<tr>
<td>(B)</td>
<td>(A \Rightarrow C \quad B \Rightarrow C)</td>
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<tr>
<td>(C)</td>
<td>(C)</td>
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<tr>
<td><strong>Reductio ad absurdum</strong></td>
<td></td>
</tr>
<tr>
<td>(\neg \neg A)</td>
<td>(\frac{\neg \neg A}{A} \quad \text{RAA})</td>
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Second Example

Prove \( a \land \neg a \Rightarrow b. \)
Second Example

Prove $a \land \neg a \Rightarrow b$.

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<td>1</td>
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<td>Assume $a \land \neg a$</td>
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<tr>
<td>1</td>
<td>2</td>
<td>$a$</td>
<td>$\land E1 \ 1$</td>
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<td>1</td>
<td>2</td>
<td>( a )</td>
<td>( \land E2 \ 1 )</td>
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<tr>
<td>1</td>
<td>3</td>
<td>( \neg a )</td>
<td>( \Rightarrow E \ 2,3 )</td>
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<tr>
<td>1</td>
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<td>$\land E 1$</td>
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<td>$\land E 2$</td>
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<td>1</td>
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<td>Therefore $a \land \neg a \Rightarrow b$</td>
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Prove $\neg A$ in the environment $\neg (A \lor B)$

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<td>A ( \lor B )</td>
<td>( \lor / 1 ) 1</td>
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<td>Therefore ( \neg A )</td>
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Fourth exemple (example 3.1.12)

Prove $\neg A \lor B$ in the environment $A \Rightarrow B$.

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<td>Assume $\neg (\neg A \lor B)$</td>
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<td>Assume ( \neg (\neg A \lor B) )</td>
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<td>2</td>
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<td>Therefore ( \neg A )</td>
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<td>1</td>
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<tr>
<td>1</td>
<td>8</td>
<td>$\bot$</td>
<td>$\Rightarrow E\ 1, 7$</td>
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<td>1</td>
<td>9</td>
<td>Therefore $\neg \neg (\neg A \lor B)$</td>
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<tr>
<td>1</td>
<td>7</td>
<td>$\neg A \lor B$</td>
<td>$\lor I \ 1 \ 6$</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>$\bot$</td>
<td>$\Rightarrow E \ 1, \ 7$</td>
</tr>
<tr>
<td>1</td>
<td>9</td>
<td>Therefore $\neg(\neg A \lor B)$</td>
<td>$\Rightarrow I \ 1, \ 8$</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>$\neg A \lor B$</td>
<td>$RAA \ 9$</td>
</tr>
</tbody>
</table>
Tree (example 3.1.12)

Give the tree representation of the previous proof:
Tree (example 3.1.12)

Give the tree representation of the previous proof:

\[
\begin{align*}
(1) \neg (\neg A \lor B) & \\
(3) B & \Rightarrow E \\
(4) \neg A \lor B & \lor 2 \\
(5) \bot & \Rightarrow I[2] \\
(6) \neg A & \lor 1 \\
(7) \neg A \lor B & \Rightarrow E \\
(8) \bot & \Rightarrow I[1] \\
(9) \neg \neg (\neg A \lor B) & \Rightarrow I[1] \\
(10) \neg A \lor B & RAA \\
\end{align*}
\]
Tree (example 3.1.12)

Give the tree representation of the previous proof:

\[
\begin{array}{c}
(i) A \Rightarrow B \\
(2) \neg A \\
(3) B \\
(4) \neg A \lor B \\
(5) \bot \\
(6) \neg A \\
(7) \neg A \lor B \\
(8) \bot \\
(9) \neg \neg (\neg A \lor B) \\
(10) \neg A \lor B \\
\Rightarrow \quad I[1] \\
\quad RAA \\
\Rightarrow \quad I[2] \\
\Rightarrow E \\
\end{array}
\]

\[
\begin{array}{c}
\Rightarrow \quad \lor 1 \\
\Rightarrow \quad \lor 2 \\
\Rightarrow E \\
\end{array}
\]

The environment consists of formulae occurring at non-removed leaves.
The homework example

<table>
<thead>
<tr>
<th>context</th>
<th>number</th>
<th>proof</th>
<th>justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>Assume ((p \Rightarrow \neg j) \land (\neg p \Rightarrow j) \land (j \Rightarrow m))</td>
<td>(\land E 1)</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>(\neg p \Rightarrow j)</td>
<td>(\land E 1)</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>(j \Rightarrow m)</td>
<td>(\land E 1)</td>
</tr>
<tr>
<td>1,4</td>
<td>4</td>
<td>Assume (\neg (m \lor p))</td>
<td></td>
</tr>
<tr>
<td>1,4,5</td>
<td>5</td>
<td>Assume (p)</td>
<td>(\lor I 5)</td>
</tr>
<tr>
<td>1,4,5</td>
<td>6</td>
<td>(m \lor p)</td>
<td>(\Rightarrow E 4,6)</td>
</tr>
<tr>
<td>1,4</td>
<td>7</td>
<td>(\bot)</td>
<td>(\Rightarrow I 5,7)</td>
</tr>
<tr>
<td>1,4</td>
<td>8</td>
<td>Therefore (\neg p)</td>
<td></td>
</tr>
<tr>
<td>1,4</td>
<td>9</td>
<td>(j)</td>
<td>(\Rightarrow E 2, 8)</td>
</tr>
<tr>
<td>1,4</td>
<td>10</td>
<td>(m)</td>
<td>(\Rightarrow E 3, 9)</td>
</tr>
<tr>
<td>1,4</td>
<td>11</td>
<td>(m \lor p)</td>
<td>(\lor I 10)</td>
</tr>
<tr>
<td>1,4</td>
<td>12</td>
<td>(\bot)</td>
<td>(\Rightarrow E 4, 11)</td>
</tr>
<tr>
<td>1</td>
<td>13</td>
<td>Therefore (\neg\neg (m \lor p))</td>
<td>(\Rightarrow I 4, 13)</td>
</tr>
<tr>
<td>1</td>
<td>14</td>
<td>(m \lor p)</td>
<td>(RAA 13)</td>
</tr>
</tbody>
</table>
Composing proofs

Useful: proofs from an environment can be composed together.

Admissible Cut rule:

\[
\text{If } \Gamma \vdash A \text{ and } \Gamma, A \vdash B \text{ then } \Gamma \vdash B
\]
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\]

Equivalently:

\[
\text{If } \Gamma, \Gamma' \vdash A \text{ and } \Gamma, A, \Gamma'' \vdash B \text{ then } \Gamma, \Gamma', \Gamma'' \vdash B
\]
Composing proofs

Useful: proofs from an environment can be composed together.

Admissible Cut rule:

\[ \text{If } \Gamma \vdash A \text{ and } \Gamma, A \vdash B \text{ then } \Gamma \vdash B \]

Equivalently:

\[ \text{If } \Gamma, \Gamma' \vdash A \text{ and } \Gamma, A, \Gamma'' \vdash B \text{ then } \Gamma, \Gamma', \Gamma'' \vdash B \]

Justification – we can glue proofs together:

\[ \text{If } \Gamma \vdash P : A \text{ and } \Gamma, A \vdash Q : B \text{ then } \Gamma \vdash (P, Q) : B \]
Plan

Correctness

Completeness

Tactics

Conclusion
Theorem 3.3.1

If a formula $A$ is deduced from an environment $\Gamma$ ($\Gamma \vdash A$) then $A$ is a consequence of $\Gamma$ ($\Gamma \models A$).

Every proof written in an environment $\Gamma$ is correct!
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Proof by induction on the number of lines in the proof $P$:

- Let $H_i$ be the context and $C_i$ the conclusion of the $i^{th}$ line in $P$.
- We show that for every $k$ we have $\Gamma, H_k \models C_k$. 
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- Let $H_i$ be the context and $C_i$ the conclusion of the $i^{th}$ line in $P$.
- We show that for every $k$ we have $\Gamma, H_k \models C_k$.

Hence, for the last line $n$ of the proof: $\Gamma \models A$
($H_n$ is empty and $C_n = A$)
Base case

Assume that $A$ is derived from $\Gamma$ by an empty proof.

That is, $A$ is a member of $\Gamma$.

Hence $\Gamma \models A$. 
Induction hypothesis

Assume that for every line $i < k$ of the proof we have $\Gamma, H_i \models C_i$.

Let us prove that $\Gamma, H_k \models C_k$. 
Induction hypothesis

Assume that for every line $i < k$ of the proof we have $\Gamma, H_i \models C_i$.

Let us prove that $\Gamma, H_k \models C_k$.

Three possible cases:

- Line $k$ is “Assume $C_k$”.
- Line $k$ is “Therefore $C_k$”.
- Line $k$ is “$C_k$”.
Line $k$ is “Assume $C_k$”

The formula $C_k$ is the last formula of $H_k$.

Then $\Gamma, H_k \models C_k$. 
The line $k$ is “Therefore $C_k$”

$C_k$ is the formula $B \Rightarrow D$ where:

- $B$ is the last formula of $H_{k-1}$ and $H_{k-1} = H_k, B$
- $D$ is usable on the previous line.
The line $k$ is “Therefore $C_k$”

$C_k$ is the formula $B \Rightarrow D$ where:

- $B$ is the last formula of $H_{k-1}$ and $H_{k-1} = H_k, B$
- $D$ is usable on the previous line.

Hence there exists a line $i < k$ such that $D = C_i$ and $H_i$ is a prefix of $H_{k-1}$.

By induction hypothesis, $\Gamma, H_i \models D$. 

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The line \( k \) is “Therefore \( C_k \)"

\( C_k \) is the formula \( B \Rightarrow D \) where:

- \( B \) is the last formula of \( H_{k-1} \) and \( H_{k-1} = H_k, B \)
- \( D \) is usable on the previous line.

Hence there exists a line \( i < k \) such that \( D = C_i \) and \( H_i \) is a prefix of \( H_{k-1} \).

By induction hypothesis, \( \Gamma, H_i \models D \).

Since \( H_i \) is a prefix of \( H_{k-1} \), we have \( \Gamma, H_{k-1} \models D \)

which can also be written \( \Gamma, H_k, B \models D \).

Therefore \( \Gamma, H_k \models B \Rightarrow D \).
Line $k$ is “$C_k$”

$C_k$ is then the conclusion of a rule, whose premises either:

- are usable on the previous line
- or belong to $\Gamma$. 

We only consider the rule $\land I$, the other cases being similar.

Let $D \land E$ be the premises of the rule. By induction hypothesis, we have:

$$\Gamma, H_{k-1} \models D \land E.$$

Since the line $k$ does not change the hypotheses, we have $H_k = H_{k-1}$.

Finally $D, E \models D \land E$. Transitively, $\Gamma, H_k \models C_k$.

For the other rules, it is the same proof, you just have to prove that the conclusion is a consequence of the premises.
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By induction hypothesis, we have:

$\Gamma, H_{k-1} \models D$ and $\Gamma, H_{k-1} \models E$.

Since the line $k$ does not change the hypotheses, we have $H_{k-1} = H_k$. 
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By induction hypothesis, we have:

$\Gamma, \ H_{k-1} \models D$ and $\Gamma, \ H_{k-1} \models E$.

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Finally $D, \ E \models D \land E$. Transitivity, $\Gamma, \ H_k \models C_k$.

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We prove the completeness of the rules only for formulas containing the following logic symbols: $\bot$, $\land$, $\lor$, $\Rightarrow$.

This is enough because additional symbols $\top$, $\neg$ and $\leftrightarrow$ can be regarded as abbreviations.
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**Theorem 3.4.1**

Let $\Gamma$ be a finite set of formulae and $A$ a formula. If $\Gamma \models A$ then $\Gamma \vdash A$. 
Definitions

A literal is either a variable $x$ or an implication $x \Rightarrow \bot$. $x$ and $x \Rightarrow \bot$ (abbreviated as $\neg x$) are complementary literals.
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We define a measure $m$ of formulae and of lists of formulae as:

- $m(\bot) = 0$
- $m(x) = 1$
- $m(A \Rightarrow B) = 1 + m(A) + m(B)$
- $m(A \land B) = 1 + m(A) + m(B)$
- $m(A \lor B) = 2 + m(A) + m(B)$
- $m(\Gamma) = \sum_{A \in \Gamma} m(A)$
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- $m(A \lor B) = 2 + m(A) + m(B)$
- $m(\Gamma) = \sum_{A \in \Gamma} m(A)$

For example, let $A = (a \lor \neg a)$.
$m(\neg a) = 2$, $m(A) = 5$ and $m(A, (b \land b), A) = 13$. 
Induction

We define $P(n)$ to be the following property:
for every $\Gamma$ and $A$, if $m(\Gamma, A) = n$, then if $\Gamma \models A$ then $\Gamma \vdash A$. 
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for every $\Gamma$ and $A$, if $m(\Gamma, A) = n$, then if $\Gamma \models A$ then $\Gamma \vdash A$.

To show that $P(n)$ holds for every integer $n$, we use “strong” induction (on the measure $m(\_,\_)$):

if for every $i < k$ property $P(i)$ holds, then $P(k)$ holds.
Induction

We define $P(n)$ to be the following property:
for every $\Gamma$ and $A$, if $m(\Gamma, A) = n$, then if $\Gamma \models A$ then $\Gamma \vdash A$.

To show that $P(n)$ holds for every integer $n$, we use “strong” induction (on the measure $m(\_,\_)$):

if for every $i < k$ property $P(i)$ holds, then $P(k)$ holds.

Assume that for every $i < k$, property $P(i)$ holds.
Assume that $m(\Gamma, A) = k$ and $\Gamma \models A$.
Let us show that $\Gamma \vdash A$. 
Decomposition

Idea: we decompose $\Gamma, A$ in order to apply the induction hypothesis.
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$A$ is undecomposable if $A$ is $\bot$ or a variable.
$\Gamma$ is undecomposable if $\Gamma$ is a list of literals or contain the formula $\bot$. 

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We study three cases:

Case 1: Neither $A$ nor $\Gamma$ is decomposable.
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Case 2: $A$ is decomposable.
  We decompose $A$ into two sub-formulae $B$ and $C$.
  We obtain $m(\Gamma, B) < m(\Gamma, A)$ and $m(\Gamma, C) < m(\Gamma, A)$. 

Case 3: $\Gamma$ is decomposable.
We choose a decomposable formula (other than $x \Rightarrow \bot$) in $\Gamma$. 

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   We choose a decomposable formula (other than $x \Rightarrow \bot$) in $\Gamma$. 
Case 1: neither $A$, nor $\Gamma$ are decomposable

Then:

- $\Gamma$ is a list of literals or contains the formula $\bot$.
- $A$ is $\bot$ or a variable.
Case 1: neither $A$, nor $\Gamma$ are decomposable

Then:

- $\Gamma$ is a list of literals or contains the formula $\bot$.
- $A$ is $\bot$ or a variable.

(a) If $\bot \in \Gamma$ then $A$ can be derived from $\bot$ by the rule $Efq$. 
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(a) If $\bot \in \Gamma$ then $A$ can be derived from $\bot$ by the rule $Efq$.
(b) If $\Gamma$ is a list of literals then we have two cases:
Case 1: neither \( A \), nor \( \Gamma \) are decomposable

Then:

- \( \Gamma \) is a list of literals or contains the formula \( \bot \).
- \( A \) is \( \bot \) or a variable.

(a) If \( \bot \in \Gamma \) then \( A \) can be derived from \( \bot \) by the rule \( Efq \).

(b) If \( \Gamma \) is a list of literals then we have two cases:
  - \( A = \bot \).
    Since \( s(\Gamma) \models A \), there are two complementary literals in \( \Gamma \).
    Therefore \( A \) can be derived from \( \Gamma \) by the rule \( \Rightarrow E \).
Case 1: neither $A$, nor $\Gamma$ are decomposable

Then:

- $\Gamma$ is a list of literals or contains the formula $\bot$.
- $A$ is $\bot$ or a variable.

(a) If $\bot \in \Gamma$ then $A$ can be derived from $\bot$ by the rule $Efq$.
(b) If $\Gamma$ is a list of literals then we have two cases:
  - $A = \bot$.
    Since $s(\Gamma) \models A$, there are two complementary literals in $\Gamma$.
    Therefore $A$ can be derived from $\Gamma$ by the rule $\Rightarrow E$.
  - $A$ is a variable.
    Since $\Gamma \models A$:

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     Since $s(\Gamma) \models A$, there are two complementary literals in $\Gamma$.
     Therefore $A$ can be derived from $\Gamma$ by the rule $\Rightarrow E$.

   - $A$ is a variable.
     Since $\Gamma \models A$:
     - either $\Gamma$ contains two complementary literals, and similarly $\Gamma \vdash A$
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(b) If $\Gamma$ is a list of literals then we have two cases:
  - $A = \bot$.
    Since $s(\Gamma) \models A$, there are two complementary literals in $\Gamma$. Therefore $A$ can be derived from $\Gamma$ by the rule $\Rightarrow E$.
  - $A$ is a variable.
    Since $\Gamma \models A$:
    - either $\Gamma$ contains two complementary literals, and similarly $\Gamma \vdash A$
    - or $A \in \Gamma$ and in this case $\Gamma \vdash A$ (by empty proof).
Case 2: $A$ is decomposable into $B$ and $C$

$A$ is decomposed into $B \land C$, $B \lor C$, or $B \implies C$.

We only study the case $A = B \land C$, the other cases are similar.
Case 2: $A$ is decomposable into $B$ and $C$

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We only study the case $A = B \land C$, the other cases are similar.

Since $\Gamma \models A$ and $A = B \land C$, we have $\Gamma \models B$ and $\Gamma \models C$. 
Case 2: $A$ is decomposable into $B$ and $C$

$A$ is decomposed into $B \land C$, $B \lor C$, or $B \Rightarrow C$.

We only study the case $A = B \land C$, the other cases are similar.

Since $\Gamma \models A$ and $A = B \land C$, we have $\Gamma \models B$ and $\Gamma \models C$.

The measures of $B$ and $C$ are strictly less than the measure of $A$, hence $m(\Gamma, B) < k$ and $m(\Gamma, C) < k$. 
Case 2: \( A \) is decomposable into \( B \) and \( C \)

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Since \( \Gamma \models A \) and \( A = B \land C \), we have \( \Gamma \models B \) and \( \Gamma \models C \).

The measures of \( B \) and \( C \) are strictly less than the measure of \( A \), hence \( m(\Gamma, B) < k \) and \( m(\Gamma, C) < k \).

By induction hypothesis, there exist two proofs \( P \) and \( Q \) such that \( \Gamma \vdash P : B \) and \( \Gamma \vdash Q : C \).
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$A$ is decomposed into $B \land C$, $B \lor C$, or $B \Rightarrow C$.

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The measures of $B$ and $C$ are strictly less than the measure of $A$, hence $m(\Gamma, B) < k$ and $m(\Gamma, C) < k$.
By induction hypothesis, there exist two proofs $P$ and $Q$ such that $\Gamma \vdash P : B$ and $\Gamma \vdash Q : C$.

Hence the proof “$P, Q, A$” is a proof of $A$ in the environment $\Gamma$. 
Case 3: $\Gamma$ is decomposable

There is a decomposable formula in $\Gamma$ which is either:

- $B \land C$
- $B \lor C$
- $B \Rightarrow C$ where $C \neq \bot$
- $(B \land C) \Rightarrow \bot$
- $(B \lor C) \Rightarrow \bot$
- $(B \Rightarrow C) \Rightarrow \bot$

We only study the first case.
Γ is a permutation of the list \((B \land C), \Delta\)
\( \Gamma \) is a permutation of the list \((B \land C), \Delta\).

\( \Gamma \) and \((B \land C), \Delta\) have the same measure.
Γ is a permutation of the list \((B \land C), \Delta\)

Γ and \((B \land C), \Delta\) have the same measure.

Since \(\Gamma \models A\), we have \(B, C, \Delta \models A\).
Γ is a permutation of the list \((B \land C), \Delta\)

Γ and \((B \land C), \Delta\) have the same measure.

Since \(\Gamma \models A\), we have \(B, C, \Delta \models A\).

The sum of the measures of \(B\) and \(C\) is strictly less than the measure of \(B \land C\).
Γ is a permutation of the list \((B \land C), \Delta\)

Γ and \((B \land C), \Delta\) have the same measure.

Since \(\Gamma \models A\), we have \(B, C, \Delta \models A\).

The sum of the measures of \(B\) and \(C\) is strictly less than the measure of \(B \land C\).

Hence \(m(B, C, \Delta, A) < m((B \land C), \Delta, A) = m(\Gamma, A) = k\), by induction hypothesis, there exist a proof \(P\) such that \(B, C, \Delta \vdash P : A\).
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Γ and \((B \land C), \Delta\) have the same measure.

Since \(\Gamma \models A\), we have \(B, C, \Delta \models A\).

The sum of the measures of \(B\) and \(C\) is strictly less than the measure of \(B \land C\).

Hence \(m(B, C, \Delta, A) < m((B \land C), \Delta, A) = m(\Gamma, A) = k\), by induction hypothesis, there exist a proof \(P\) such that \(B, C, \Delta \vdash P : A\).

Since \(B\) can be derived from \((B \land C)\) by the rule \(\land E1\) and \(C\) can be derived from \((B \land C)\) by the rule \(\land E2\) : “\(B, C, P\)” is a proof of \(A\) in the environment \(\Gamma\).
Plan

Correctness

Completeness

Tactics

Conclusion
Remark 3.4.2

The proof of completeness is constructive, that is it provides an algorithm to build a proof of a formula in an environment.

However, this algorithm can lead to long proofs.
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However, this algorithm can lead to long proofs.

The tool http://teachinglogic.univ-grenoble-alpes.fr/DN/ builds proofs more more efficiently.
Remark 3.4.2

The proof of completeness is constructive, that is it provides an algorithm to build a proof of a formula in an environment.

However, this algorithm can lead to long proofs.


It uses “optimised” tactics presented in section 3.2.

For example, to prove $B \lor C$:

- First try to prove $B$
- If failure, then try to prove $C$
- Otherwise, use tactic 10 (prove $C$ under the hypothesis $\neg B$)
Proof tactics

We wish to prove $A$ in the environment $\Gamma$

The 13 following tactics must be used in the following order!

▶ Tactics 1 to 3 : the proof is over
▶ Tactics 4 to 6 : proof guided by the conclusion (Intro rules)
▶ Tactics 7 to 9 : proof guided by the environment (Elim rules)
▶ Tactics 10 to 13 : reasoning by absurd
Tactic 1

If $A \in \Gamma$ then the empty proof is obtained.
Tactic 2

If \( A \) is the consequence of a rule whose premises are in \( \Gamma \), then the obtained proof is “\( A \)”. 
Tactic 3

If Γ contains a contradiction, that is a formula $B$ and a formula $\neg B$, then the obtained proof is “$\bot, A$".
Tactic 4

If $A$ is $B \land C$ then:

<table>
<thead>
<tr>
<th>contexte</th>
<th>preuve</th>
<th>justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma$</td>
<td>$B$</td>
<td>$\cdots P \cdots$</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>$C$</td>
<td>$\cdots Q \cdots$</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>$B \land C$</td>
<td>$\land I$</td>
</tr>
</tbody>
</table>

The proofs can fail (if $\Gamma \not\models A$). Here, if the proof of $B$ or $C$ fails, the proof of $A$ fails too. In the remainder of the lecture, we do not highlight the failure cases anymore, unless another proof has to be tried.
Tactic 4

If $A$ is $B \land C$ then:

<table>
<thead>
<tr>
<th>contexte</th>
<th>preuve</th>
<th>justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma$</td>
<td>$B$</td>
<td>$\cdots P \cdots$</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>$C$</td>
<td>$\cdots Q \cdots$</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>$B \land C$</td>
<td>$\land I$</td>
</tr>
</tbody>
</table>

The proofs can fail (if $\Gamma \not\models A$).
Here, if the proof of $B$ or $C$ fails, the proof of $A$ fails too.
In the remainder of the lecture, we do not highlight the failure cases anymore, unless another proof has to be tried.
Tactic 5 (mandatory)

If \( A \) is \( B \Rightarrow C \), then prove \( C \) under hypothesis \( B \):

<table>
<thead>
<tr>
<th>contexte</th>
<th>preuve</th>
<th>justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma, B )</td>
<td>Assume ( B )</td>
<td>( \cdots P \cdots )</td>
</tr>
<tr>
<td>( \Gamma, B )</td>
<td>( C )</td>
<td>( \Rightarrow I )</td>
</tr>
<tr>
<td>( \Gamma )</td>
<td>Therefore ( B \Rightarrow C )</td>
<td></td>
</tr>
</tbody>
</table>
Tactic 6

If $A$ is $B \lor C$, then prove $B$:

<table>
<thead>
<tr>
<th>contexte</th>
<th>preuve</th>
<th>justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma$</td>
<td>$B$</td>
<td>$\cdots P \cdots$</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>$B \lor C$</td>
<td>$\lor I_1$</td>
</tr>
</tbody>
</table>

If the proof of $B$ fails then prove $C$:

<table>
<thead>
<tr>
<th>contexte</th>
<th>preuve</th>
<th>justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma$</td>
<td>$C$</td>
<td>$\cdots P \cdots$</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>$B \lor C$</td>
<td>$\lor I_2$</td>
</tr>
</tbody>
</table>

If the proof of $C$ fails, try the following tactics.
Tactic 7

If $B \land C$ is in the environment, then prove $A$ starting from formulae $B$, $C$, replacing $B \land C$ in the environment:

<table>
<thead>
<tr>
<th>contexte</th>
<th>preuve</th>
<th>justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma, B \land C$</td>
<td>$B$</td>
<td>$\land E1$</td>
</tr>
<tr>
<td>$\Gamma, B \land C$</td>
<td>$C$</td>
<td>$\land E2$</td>
</tr>
<tr>
<td>$\Gamma, B \land C$</td>
<td>$A$</td>
<td>$\cdots P \cdots$</td>
</tr>
</tbody>
</table>
Tactic 8

If \( B \lor C \) is in the environment, then:

- prove \( A \) in the environment where \( B \) replaces \( B \lor C \).
- prove \( A \) in the environment where \( C \) replaces \( B \lor C \).

<table>
<thead>
<tr>
<th>contexte</th>
<th>preuve</th>
<th>justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma, B \lor C, B )</td>
<td>Assume ( B )</td>
<td>( \cdots P \cdots )</td>
</tr>
<tr>
<td>( \Gamma, B \lor C, B )</td>
<td>( A )</td>
<td>( \Rightarrow I )</td>
</tr>
<tr>
<td>( \Gamma, B \lor C )</td>
<td>Therefore ( B \Rightarrow A )</td>
<td>( \cdots Q \cdots )</td>
</tr>
<tr>
<td>( \Gamma, B \lor C, C )</td>
<td>Assume ( C )</td>
<td>( \Rightarrow I )</td>
</tr>
<tr>
<td>( \Gamma, B \lor C, C )</td>
<td>( A )</td>
<td>( \lor E )</td>
</tr>
<tr>
<td>( \Gamma, B \lor C )</td>
<td>Therefore ( C \Rightarrow A )</td>
<td></td>
</tr>
<tr>
<td>( \Gamma, B \lor C )</td>
<td>( A )</td>
<td></td>
</tr>
</tbody>
</table>
Tactic 9

If $\neg(B \lor C)$ is in the environment, then

- we derive $\neg B$ by the proof $P4$ and
- $\neg C$ by the proof $P5$ (proofs requested in exercise 59).
- Let $P$ the proof of $A$ in the environment where $\neg B$, $\neg C$ replace the formula $\neg(B \lor C)$.

The proof of $A$ is “$P4, P5, P$”.
Tactic 10

If $A$ is $B \lor C$, then prove $C$ under hypothesis $\neg B$: let $P$ the obtained proof.

"Assume $\neg B$, $P$, Therefore $\neg B \Rightarrow C$" is a proof of the formula $\neg B \Rightarrow C$ which is equivalent to $A$.

To obtain the proof of $A$, simply add the proof $P1$, requested in exercise 59 of $A$ in the environment $\neg B \Rightarrow C$.

The proof obtained from $A$ is therefore "Assume $\neg B$, $P$, Therefore $\neg B \Rightarrow C$, $P1$".
Tactic 11

If \( \neg (B \land C) \) is in the environment, first we deduce from it \( \neg B \lor \neg C \) (by the proof \( P3 \) requested in exercise 59).

Then we reason case by case, using \( \lor E \), as follows:

1. prove \( A \) in the environment where \( \neg B \) replaces \( \neg (B \land C) \): Let \( P \) be the obtained proof,

2. prove \( A \) in the environment where \( \neg C \) replaces \( \neg (B \land C) \): Let \( Q \) be the obtained proof.

The proof of \( A \) is “\( P3, \) Assume \( \neg B, P \), Therefore \( \neg B \Rightarrow A \), Assume \( \neg C, Q \), Therefore \( \neg C \Rightarrow A, A \)”.
Tactic 12

If $\neg (B \Rightarrow C)$ is in the environment, then

- we derive $B$ by the proof $P_6$,
- $\neg C$ by the proof $P_7$ (proofs requested in exercise 59).
- Let $P$ be the proof of $A$ in the environment where $B$, $\neg C$ replace the formula $\neg (B \Rightarrow C)$.

The proof of $A$ is “$P_6, P_7, P$”.
Tactic 13

If $B \Rightarrow C$ is in the environment and if $C \neq \perp$, i.e. if $B \Rightarrow C$ is not $\neg B$, first we derive $\neg B \lor C$ in the environment $B \Rightarrow C$ (by proof $P2$ from exercise 59).

We then we reason by cases, using $\lor E$:

- prove $A$ in the environment where $\neg B$ replaces $B \Rightarrow C$: Let $P$ the obtained proof,

- prove $A$ in the environment where $C$ replaces $B \Rightarrow C$: Let $Q$ the obtained proof.

The proof of $A$ is “$P2$, Assume $\neg B$, $P$, Therefore $\neg B \Rightarrow A$, Assume $C$, $Q$, Therefore $C \Rightarrow A$, $A$”.
Example

Proof of Peirce’s formula:

\[((p \Rightarrow q) \Rightarrow p) \Rightarrow p\]
Proof plan

Tactic 5 is mandatory!

Proof $Q$:
Assume $(p \Rightarrow q) \Rightarrow p$

$Q_1$ proof or $p$ in the environment $(p \Rightarrow q) \Rightarrow p$

Therefore $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$

Proof $Q_1$ necessarily uses tactic 13 (as the environment is $B \Rightarrow C = (p \Rightarrow q) \Rightarrow p$).

Therefore (in order to use $\lor E$) we have to prove $p$, both:

- in the environment $\neg B = \neg (p \Rightarrow q)$
- in the environment $C = p$.
Plan of the proof of $Q_1$

Proof $Q_1$:

$Q_{11}$ is the proof of $\neg B \lor C$ in the environment $B \Rightarrow C$, see exercise 59

Assume $\neg (p \Rightarrow q)$

$Q_{12}$ proof of $p$ in the environment $\neg (p \Rightarrow q)$

Therefore $\neg (p \Rightarrow q) \Rightarrow p$

Assume $p$

$Q_{13}$ proof of $p$ in the environment $p$

Therefore $p \Rightarrow p$

$p$
Proof of $Q_1$

$Q_{13}$ is the empty proof, since $A = C = p$.

$Q_{12}$ is the proof of $C = p$ in the environment $\neg(p \Rightarrow q)$. Since $\neg A$ is an abbreviation of $A \Rightarrow \bot$, this proof is the proof $P_6$ requested in exercise 59, where $B = p$ and $C = q$.

By gluing pieces $Q_1$, $Q_{11}$, $Q_{12}$, $Q_{13}$, we obtain the proof $Q$.

The proof $Q_{12}$ can also be made without using the tactics.
Plan

Correctness

Completeness

Tactics

Conclusion
Today

- Propositional Natural Deduction is correct and complete.
- Tactics for building a proof
Automated proofs

To automatically obtain the proofs in the system, we recommend the following software (implementing the 13 previous tactics):

http://teachinglogic.univ-grenoble-alpes.fr/DN/

People who prefer tree-like proofs can use the following software:

http://www-sop.inria.fr/marelle/Laurent.Thery/peanoware/Nd.html
Overview of the Semester

TODAY

- Propositional logic
- Propositional resolution
- Natural deduction for propositional logic *
- First order logic

MIDTERM EXAM

- Logical basis for automated proving ("first-order resolution")
- First-order natural deduction

EXAM