Natural Deduction

Properties and tactics

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Last lecture

Natural deduction

- Rules
- Context
- Proofs

Reminder of the rules

	Introduction	Elimination
Implication	$\dfrac{\begin{array}{c}[A]\\ \dots\\ B\end{array}}{A \Rightarrow B}\ \Rightarrow I$	$\dfrac{A \quad A \Rightarrow B}{B}\ \Rightarrow E$
Conjunction	$\dfrac{A \quad B}{A \wedge B}\ \wedge I$	$\dfrac{A \wedge B}{A}\ \wedge E1$ $\dfrac{A \wedge B}{B}\ \wedge E2$
Disjunction	$\dfrac{A}{A \vee B}\ \vee I1$ $\dfrac{B}{A \vee B}\ \vee I2$	$\dfrac{A \vee B \quad A \Rightarrow C \quad B \Rightarrow C}{C}\ \vee E$
	Ex falso quodlibet	
$\bot$	$\dfrac{\bot}{A}\ Efq$	
	Reductio ad absurdum	
	$\dfrac{\neg\neg A}{A}\ RAA$	

Third Example: with an environment

Prove $\neg A$ in the environment $\neg(A \vee B)$

environment			
reference		formula	
(i)		$\neg(A \vee B)$	
context	number	proof	justification
1	1	`Assume` A	
1	2	$A \vee B$	$\vee I1$ 1
1	3	$\perp$	$\Rightarrow E$ i,2
	4	`Therefore` $\neg A$	$\Rightarrow I$ 1,3

A proof with a disjunctive hypothesis

Prove that $\neg A \vee \neg B \Rightarrow \neg(A \wedge B)$.

context	number	proof	justification
1	1	`Assume` $\neg A \vee \neg B$	
1,2	2	`Assume` $A \wedge B$	
1,2	3	A	$\wedge E$ 2
1,2	4	B	$\wedge E$ 2
1,2,5	5	`Assume` $\neg A$	
1,2,5	6	$\perp$	$\Rightarrow E$ 3,5
1,2	7	`Therefore` $\neg A \Rightarrow \perp$	$\Rightarrow I$ 5,6
1,2,8	8	`Assume` $\neg B$	
1,2,8	9	$\perp$	$\Rightarrow E$ 4,8
1,2	10	`Therefore` $\neg B \Rightarrow \perp$	$\Rightarrow I$ 8,9
1,2	11	$\perp$	$\vee E$ 1,7,10
1	12	`Therefore` $\neg(A \wedge B)$	$\Rightarrow I$ 2,11
	13	`Therefore` $\neg A \vee \neg B \Rightarrow \neg(A \wedge B)$	$\Rightarrow I$ 1,12

Fourth exemple (example 3.1.12)

Prove $\neg A \vee B$ in the environment $A \Rightarrow B$.

environment			
reference		formula	
(i)		$A \Rightarrow B$	
context	number	proof	justification
1	1	`Assume` $\neg(\neg A \vee B)$	
1,2	2	`Assume` A	
1,2	3	B	$\Rightarrow E$ i, 2
1,2	4	$\neg A \vee B$	$\vee I2$ 3
1,2	5	$\perp$	$\Rightarrow E$ 1, 4
1	6	`Therefore` $\neg A$	$\Rightarrow I$ 2, 5
1	7	$\neg A \vee B$	$\vee I1$ 6
1	8	$\perp$	$\Rightarrow E$ 1, 7
	9	`Therefore` $\neg\neg(\neg A \vee B)$	$\Rightarrow I$ 1, 8
	10	$\neg A \vee B$	RAA 9

Tree (example 3.1.12)

Give the tree representation of the previous proof:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(1)\cancel{\neg(\neg A \vee B)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{(i)A \Rightarrow B \quad (2)\cancel{A}}{(3)B}\Rightarrow E}{(4)\neg A \vee B}\vee 2}{(5)\bot}\Rightarrow E}{(6)\neg A}\Rightarrow I[2]}{(7)\neg A \vee B}\vee 1}{(8)\bot}\Rightarrow E}{(9)\neg\neg(\neg A \vee B)}\Rightarrow I[1]}{(10)\neg A \vee B}RAA
$$

The environment consists of formulas occurring at non-removed leaves.

Intuitionism and constructivism (Brouwer, 1881-1966)

In the wake of Poincaré, he founded (in 1918) the **intuitionist** philosophy: the validity of mathematics should be verifiable by the human mind.

- refusal of infinite objects such as the ones of set theory
- in particular, notion of **constructible real number** = algorithm that produces its digits

Example of a non-constructive proof : assume $P(0)$ and $\neg P(2)$.
Then $\exists x(P(x) \wedge \neg P(x+1))$ ... but we don't know whether $x = 0$ or $x = 1$ is the "correct" witness for that property.
The introduction rules for $\vee$ make it explicit which case is true: following the reasoning step-by-step is an *algorithm*!

However the rule $\dfrac{\neg\neg A}{A}$ allows to override that constraint.

Our running example

context	number	proof	justification
1	1	`Assume` $(p \Rightarrow \neg j) \wedge (\neg p \Rightarrow j) \wedge (j \Rightarrow m)$	
1	2	$\neg p \Rightarrow j$	$\wedge E$ 1
1	3	$j \Rightarrow m$	$\wedge E$ 1
1,4	4	`Assume` $\neg(m \vee p)$	
1,4,5	5	`Assume` p	
1,4,5	6	$m \vee p$	$\vee I$ 5
1,4,5	7	$\perp$	$\Rightarrow E$ 4,6
1,4	8	`Therefore` $\neg p$	$\Rightarrow I$ 5,7
1,4	9	j	$\Rightarrow E$ 2, 8
1,4	10	m	$\Rightarrow E$ 3, 9
1,4	11	$m \vee p$	$\vee I$ 10
1,4	12	$\perp$	$\Rightarrow E$ 4, 11
1	13	`Therefore` $\neg\neg(m \vee p)$	$\Rightarrow I$ 4, 13
1	14	$m \vee p$	*RAA* 13

Theorem

Theorem 3.3.1

If a formula A is deduced from an environment Γ ($\Gamma \vdash A$) then A is a consequence of Γ ($\Gamma \models A$).

Every proof written in an environment Γ is correct!

Proof by induction on the number of lines in the proof P:

- Let H_i be the context and C_i the conclusion of the ith line in P.
- We show that for every k we have $\Gamma, H_k \models C_k$.

Hence, for the last line n of the proof: $\Gamma \models A$
(H_n is empty and $C_n = A$)

Case where line k is "`Therefore` $B \Rightarrow D$"

We note H_k the context for line k:

- the previous context is H_k, B
- D is usable on the previous line.

Hence there exists a line $i < k$ whose conclusion is D and such that H_i is a subset of H_k, B.
By induction hypothesis, $\Gamma,\ H_i \models D$.

Thus we also have $\Gamma,\ H_k,\ B \models D$.
Therefore $\Gamma,\ H_k \models B \Rightarrow D$.

Other cases

C_k is then the conclusion of a rule, whose premises either:

- are usable on the previous line
- or belong to Γ.

By induction hypothesis, the premises of the rule are consequences of Γ, H_k.

We can check that for every rule, the conclusion is a consequence of the premises.
Transitively, $\Gamma,\ H_k \models C_k$.

Theorem

Theorem 3.4.1

Let Γ be a finite set of formulas and A a formula.
If $\Gamma \models A$ then $\Gamma \vdash A$.

Outline of the proof

We decompose

- A
- or a formula in Γ

in order to apply the induction hypothesis.

For example: if $\Gamma, B \Rightarrow C \vDash A$ then $\Gamma, \neg B \vDash A$ and $\Gamma, C \vDash A$.

Subtelties used in the proof

Base cases :

- A is undecomposable if A is $\perp$ or a variable.
- Γ is undecomposable if Γ is a list of literals or contains the formula $\perp$.

- If one decomposes a formula into its subformulas they do not always get a rule of natural deduction
- sometimes this leads us to replace a formula with other, more elaborate formulas
- hence the need for a measure m over formulas (more elaborate than the number of connectives). For example $m(A \vee B) = 2 + m(A) + m(B)$.

Remark 3.4.2

The proof of completeness is constructive, that is: it provides an algorithm to build a proof of a formula in an environment.

However, this algorithm can lead to long proofs.

The tool
http://teachinglogic.univ-grenoble-alpes.fr/DN/
builds proofs more more efficiently.
It uses "optimised" tactics presented in section 3.2.

For example, to prove $B \vee C$:

- First try to prove B
- If it fails, then try to prove C
- Otherwise, use tactic 10 (prove C under the hypothesis $\neg B$)

Proof tactics

We wish to prove A in the environment Γ

The 13 following tactics must be used in the following order!

- Tactics 1 to 3 : the proof is over
- Tactics 4 to 6 : proof guided by the conclusion (Intro rules)
- Tactics 7 to 9 : proof guided by the environment (Elim rules)
- Tactics 10 to 13 : reasoning by absurd

Tactic 1 (trivial)

If $A \in \Gamma$ then the empty proof is sufficient.

Tactic 2 (trivial + intro)

If A is the consequence of a rule whose premises are in Γ, then the obtained proof is
"A".

Tactic 3 (Efq)

If Γ contains a contradiction, that is a formula B and a formula $\neg B$, then the obtained proof is "$\bot$, A".

Tactic 4

If A is $B \wedge C$ then:

context	proof	justification
Γ	B	$\cdots P \cdots$
Γ	C	$\cdots Q \cdots$
Γ	$B \wedge C$	$\wedge I$

The proofs can fail (if $\Gamma \not\models A$).
Here, if the proof of B or C fails, the proof of A fails too.

In the remainder of the lecture, we do not highlight the failure cases anymore, unless another proof has to be tried.

Tactic 5

If A is $B \Rightarrow C$, then prove C under hypothesis B:

context	proof	justification
Γ, B	`Assume` B	
Γ, B	C	$\cdots P \cdots$
Γ	`Therefore` $B \Rightarrow C$	$\Rightarrow I$

Tactic 6

If A is $B \vee C$, then prove B :

context	proof	justification
Γ	B	$\cdots P \cdots$
Γ	$B \vee C$	$\vee I1$

If the proof of B fails then prove C :

context	proof	justification
Γ	C	$\cdots P \cdots$
Γ	$B \vee C$	$\vee I2$

If the proof of C fails, try the following tactics.

Tactic 7

If $B \wedge C$ is in the environment, then prove A starting from formulas B, C, replacing $B \wedge C$ in the environment:

context	proof	justification
$\Gamma, B \wedge C$	B	$\wedge E1$
$\Gamma, B \wedge C$	C	$\wedge E2$
$\Gamma, B \wedge C$	A	$\cdots P \cdots$

Tactic 8

If $B \vee C$ is in the environment, then:

- prove A in the environment where B replaces $B \vee C$.
- prove A in the environment where C replaces $B \vee C$.

context	proof	justification
$\Gamma, B \vee C, B$	`Assume` B	
$\Gamma, B \vee C, B$	A	$\cdots P \cdots$
$\Gamma, B \vee C$	`Therefore` $B \Rightarrow A$	$\Rightarrow I$
$\Gamma, B \vee C, C$	`Assume` C	
$\Gamma, B \vee C, C$	A	$\cdots Q \cdots$
$\Gamma, B \vee C$	`Therefore` $C \Rightarrow A$	$\Rightarrow I$
$\Gamma, B \vee C$	A	$\vee E$

Tactic 9

If $\neg(B \vee C)$ is in the environment, then

- we derive $\neg B$ by the proof $P4$ and
- $\neg C$ by the proof $P5$ (proofs requested in exercise 59).
- Let P the proof of A in the environment where $\neg B$, $\neg C$ replace the formula $\neg(B \vee C)$.

The proof of A is "$P4$, $P5$, P".

Tactic 10

If A is $B \vee C$, then prove C under hypothesis $\neg B$: let P the obtained proof.
"`Assume` $\neg B$, P, `Therefore` $\neg B \Rightarrow C$" is a proof of the formula $\neg B \Rightarrow C$ which is equivalent to A.

To obtain the proof of A, simply add the proof $P1$, requested in exercise 59 of A in the environment $\neg B \Rightarrow C$.
The proof obtained from A is therefore "`Assume` $\neg B$, P, `Therefore` $\neg B \Rightarrow C$, $P1$".

Tactic 11

If $\neg(B \wedge C)$ is in the environment, then we deduce from it $\neg B \vee \neg C$ by the proof $P3$ requested in exercise 59 then we reason case by case as follows:

- prove A in the environment where $\neg B$ replaces $\neg(B \wedge C)$: Let P the obtained proof,
- prove A in the environment where $\neg C$ replaces $\neg(B \wedge C)$: Let Q the obtained proof.

The proof of A is "$P3$, `Assume` $\neg B$, P, `Therefore` $\neg B \Rightarrow A$, `Assume` $\neg C$, Q, `Therefore` $\neg C \Rightarrow A$, A".

Tactique 12

If $\neg(B \Rightarrow C)$ is in the environment, then

- we derive B by the proof $P6$,
- $\neg C$ by the proof $P7$ (proofs requested in exercise 59).
- Let P the proof of A in the environment where B, $\neg C$ replace the formula $\neg(B \Rightarrow C)$.

The proof of A is "$P6$, $P7$, P".

Tactic 13

If $B \Rightarrow C$ is in the environment and if $C \neq \bot$, i.e. if $B \Rightarrow C$ is not $\neg B$, then,
we derive $\neg B \vee C$ in the environment $B \Rightarrow C$ by proof $P2$ from exercise 59, then we reason by cases:

- prove A in the environment where $\neg B$ replaces $B \Rightarrow C$: Let P the obtained proof,
- prove A in the environment where C replaces $B \Rightarrow C$: Let Q the obtained proof.

The proof of A is "$P2$, `Assume` $\neg B$, P, `Therefore` $\neg B \Rightarrow A$, `Assume` C, Q, `Therefore` $C \Rightarrow A$, A".

Example

Proof of Peirce's formula:

$$((p \Rightarrow q) \Rightarrow p) \Rightarrow p$$

Proof plan

Tactic 5 is compulsory!

Proof Q:
`Assume` $(p \Rightarrow q) \Rightarrow p$
Q_1 proof or p in the environment $(p \Rightarrow q) \Rightarrow p$
`Therefore` $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$

Proof Q_1 necessarily uses tactic 13 (the environment is $B \Rightarrow C = (p \Rightarrow q) \Rightarrow p$).

Hence we have to prove p both:

- in the environment $\neg B = \neg(p \Rightarrow q)$
- in the environment $C = p$.

Plan of the proof of Q_1

Proof Q_1 :
Q_{11} is the proof of $\neg B \vee C$ in the environment $B \Rightarrow C$, see exercise 59
`Assume` $\neg(p \Rightarrow q)$
Q_{12} proof of p in the environment $\neg(p \Rightarrow q)$
`Therefore` $\neg(p \Rightarrow q) \Rightarrow p$
`Assume` p
Q_{13} proof of p in the environment p
`Therefore` $p \Rightarrow p$
p

Proof of Q_1

Q_{13} is the empty proof, since $A = C = p$.

Q_{12} is the proof of $C = p$ in the environment $\neg(p \Rightarrow q)$.
Since $\neg A$ is an abbreviation of $A \Rightarrow \bot$, this proof is the proof P_6 requested in exercise 59, where $B = p$ and $C = q$.

By gluing pieces $Q_1,\ Q_{11},\ Q_{12},\ Q_{13}$, we obtain the proof Q.

The proof Q_{12} can also be done without using the tactics.

Today

- Propositional Natural Deduction is correct and complete.
- Tactics for building a proof

Automated proofs

To automatically obtain the proofs in the system, we recommend the following software (implementing the 13 previous tactics):

http://teachinglogic.univ-grenoble-alpes.fr/DN/

Overview of the Semester

- Propositional logic
- Propositional resolution
- Natural deduction for propositional logic *
- First order logic

MIDTERM EXAM

- Logical basis for automated proving (“first-order resolution”)
- First-order natural deduction

EXAM

Quizz d'entraînement

- Chapitre 1 : https://cours.univ-grenoble-alpes.fr/mod/quiz/view.php?id=337891
- Chapitres 2 et 3 : https://cours.univ-grenoble-alpes.fr/mod/quiz/view.php?id=348521