Natural Deduction

Properties and tactics

Benjamin Wack

Université Grenoble Alpes

February 2018
Last lecture

Natural deduction

- Rules
- Context
- Proofs
## Reminder of the rules

<table>
<thead>
<tr>
<th>Introduction</th>
<th>Elimination</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Implication</strong></td>
<td><strong>Elimination</strong></td>
</tr>
<tr>
<td>( [A] )</td>
<td>( A \vdash A \Rightarrow B \Rightarrow E )</td>
</tr>
<tr>
<td>( \ldots )</td>
<td>( B )</td>
</tr>
<tr>
<td>( B \Rightarrow )</td>
<td>( \Rightarrow I )</td>
</tr>
<tr>
<td>( A \Rightarrow B \Rightarrow I )</td>
<td>( A \Rightarrow B \Rightarrow E )</td>
</tr>
<tr>
<td><strong>Conjunction</strong></td>
<td><strong>( \land E )</strong></td>
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<tr>
<td>( A \land B )</td>
<td>( A \land B )</td>
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<tr>
<td>( \land I )</td>
<td>( A \land B )</td>
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<tr>
<td>( A \land B )</td>
<td>( \land E_1 )</td>
</tr>
<tr>
<td>( A \land B )</td>
<td>( \land E_2 )</td>
</tr>
<tr>
<td><strong>Disjunction</strong></td>
<td><strong>( \lor E )</strong></td>
</tr>
<tr>
<td>( A \lor B )</td>
<td>( A \lor B )</td>
</tr>
<tr>
<td>( \lor I_1 )</td>
<td>( A \Rightarrow C \Rightarrow C \lor E )</td>
</tr>
<tr>
<td>( A \lor B )</td>
<td>( B \Rightarrow C \lor E )</td>
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<tr>
<td><strong>Ex falso quodlibet</strong></td>
<td><strong>Etq</strong></td>
</tr>
<tr>
<td>( \bot )</td>
<td>( \bot )</td>
</tr>
<tr>
<td><strong>Reductio ad absurdum</strong></td>
<td><strong>RAA</strong></td>
</tr>
<tr>
<td>( \neg \neg A )</td>
<td>( \neg \neg A )</td>
</tr>
<tr>
<td>( A )</td>
<td>( A \Rightarrow B \Rightarrow E )</td>
</tr>
</tbody>
</table>
Third Example: with an environment

Prove \( \neg A \) in the environment \( \neg (A \lor B) \)

<table>
<thead>
<tr>
<th>environment</th>
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<tbody>
<tr>
<td>reference</td>
</tr>
<tr>
<td>(i)</td>
</tr>
<tr>
<td>context</td>
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<td></td>
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<table>
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<th>justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>Assume $A$</td>
<td></td>
</tr>
</tbody>
</table>

environment

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<tr>
<td>1</td>
<td>1</td>
<td>Assume $A$</td>
<td>$\lor / 1 \ 1$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$A \lor B$</td>
<td></td>
</tr>
</tbody>
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<tr>
<td>1</td>
<td>1</td>
<td>Assume A</td>
<td>( \lor )1 1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>A \lor B</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>\bot</td>
<td>( \Rightarrow ) E i,2</td>
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<td>1</td>
<td>1</td>
<td>Assume ( A )</td>
<td>( \lor / 1 ) 1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>( A \lor B )</td>
<td>( \Rightarrow E \ i,2 )</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>( \bot )</td>
<td>( \Rightarrow I \ 1,3 )</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>Therefore ( \neg A )</td>
<td></td>
</tr>
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</table>
Fourth exemple (example 3.1.12)

Prove $\neg A \lor B$ in the environment $A \Rightarrow B$.

<table>
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<td>$(i)$</td>
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</tr>
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<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>Assume $\neg(\neg A \lor B)$</td>
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<td>1</td>
<td>1</td>
<td>Assume $\neg (\neg A \lor B)$</td>
<td></td>
</tr>
<tr>
<td>1,2</td>
<td>2</td>
<td>Assume $A$</td>
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</tr>
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<tr>
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<td>1</td>
<td>Assume $\neg (\neg A \lor B)$</td>
<td>$\Rightarrow E i, 2$</td>
</tr>
<tr>
<td>1,2</td>
<td>2</td>
<td>Assume $A$</td>
<td></td>
</tr>
<tr>
<td>1,2</td>
<td>3</td>
<td>$B$</td>
<td></td>
</tr>
</tbody>
</table>

The proof table is as follows:

- The environment contains the formula $A \Rightarrow B$.
- The proof is constructed by assuming $\neg A \lor B$ and then deriving $B$.
- The final step uses the elimination rule ($E$) with the assumption $A \Rightarrow B$ and the subproof $\neg A \lor B$.

The proof is completed by discharging the assumption $\neg A \lor B$. 

---

**Reference**

B. Wack *et al.* (UGA)
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<td>3</td>
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<td>$\Rightarrow E i, 2$</td>
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<tr>
<td>1,2</td>
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<td>$\lor l2 \ 3$</td>
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<td>1</td>
<td>Assume ( \neg(\neg A \lor B) )</td>
<td></td>
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<td>1,2</td>
<td>2</td>
<td>Assume ( A )</td>
<td>( \Rightarrow E i, 2 )</td>
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<td>3</td>
<td>( B )</td>
<td>( \lor i 2 3 )</td>
</tr>
<tr>
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<td>4</td>
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environment

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<td>( \bot )</td>
<td>( \Rightarrow I \ 2, 5 )</td>
</tr>
<tr>
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<td>Therefore ( \neg A )</td>
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<td>6</td>
<td>Therefore $\neg A$</td>
<td></td>
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<tr>
<td>1</td>
<td>7</td>
<td>$\neg A \lor B$</td>
<td>$\lor I 1 6$</td>
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<td>Assume $\neg(\neg A \lor B)$</td>
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<td>$\lor \ i2 \ 3$</td>
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<td>$B$</td>
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<td>$\lor \ i1 \ 6$</td>
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<td>1</td>
<td>8</td>
<td>$\bot$</td>
<td>$\Rightarrow E \ 1, 7$</td>
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<td>(i)</td>
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<tr>
<td>1</td>
<td>1</td>
<td>Assume ( \neg(\neg A \lor B) )</td>
<td>( \Rightarrow E \ i, 2 )</td>
</tr>
<tr>
<td>1,2</td>
<td>2</td>
<td>Assume ( A )</td>
<td>( \lor I 2 \ 3 )</td>
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<td>1,2</td>
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<td>( B )</td>
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<td>( \bot )</td>
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<td>Therefore ( \neg A )</td>
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<td>8</td>
<td>( \bot )</td>
<td></td>
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<tr>
<td>1</td>
<td>9</td>
<td>Therefore ( \neg \neg (\neg A \lor B) )</td>
<td>( \Rightarrow I 1, 8 )</td>
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Prove $\neg A \lor B$ in the environment $A \Rightarrow B$.

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<td>$\perp$</td>
<td>$\Rightarrow E 1, 7$</td>
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<tr>
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<tr>
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<td>8</td>
<td>$\perp$</td>
<td>$\Rightarrow E 1, 7$</td>
</tr>
<tr>
<td>9</td>
<td>Therefore $\neg (\neg A \lor B)$</td>
<td>$\Rightarrow I 1, 8$</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>$\neg A \lor B$</td>
<td>$\Rightarrow E 1, 7$</td>
<td></td>
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</table>

$\Rightarrow$ denotes a step where a rule is applied, $\lor I$ and $\lor E$ are the rules for disjunction introduction and elimination, respectively, and RAA stands for Reductio Ad Absurdum.
Tree (example 3.1.12)

Give the tree representation of the previous proof:
Tree (example 3.1.12)

Give the tree representation of the previous proof:

\[
\begin{array}{c}
\text{(1) } \neg(A \lor B) \\
\text{(2) } A \Rightarrow B \\
\text{(3) } B \\
\text{(4) } \neg A \lor B \\
\text{(5) } \bot \\
\text{(6) } \neg A \\
\text{(7) } \neg A \lor B \\
\text{(8) } \bot \\
\text{(9) } \neg \neg(A \lor B) \\
\text{(10) } \neg A \lor B
\end{array}
\]

\[
\frac{\text{(1) } \neg(A \lor B) \quad \text{(2) } A \Rightarrow B}{\text{(3) } B \quad \text{(4) } \neg A \lor B \quad \text{\(\lor\) 2} \quad \Rightarrow E}
\]

\[
\frac{\text{(5) } \bot \quad \Rightarrow I[2]}{\text{(6) } \neg A \quad \text{\(\lor\) 1} \quad \Rightarrow E}
\]

\[
\frac{\text{(7) } \neg A \lor B \quad \Rightarrow E}{\text{(8) } \bot \quad \Rightarrow I[1] \\
\text{(9) } \neg \neg(A \lor B) \quad \Rightarrow I[1] \quad \text{RAA}}
\]

\[
\frac{\text{(8) } \bot \quad \Rightarrow I[1]}{\text{(9) } \neg \neg(A \lor B) \quad \Rightarrow I[1] \quad \text{RAA}}
\]

\[
\frac{\text{(7) } \neg A \lor B \quad \Rightarrow E}{\text{(8) } \bot \quad \Rightarrow I[1] \\
\text{(9) } \neg \neg(A \lor B) \quad \Rightarrow I[1] \quad \text{RAA}}
\]
Tree (example 3.1.12)

Give the tree representation of the previous proof:

\[
\begin{align*}
(1) & \neg (\neg A \vee B) \\
(2) & A \Rightarrow B \\
(3) & B \\
(4) & \neg A \vee B \\
(5) & \bot \\
(6) & \neg A \\
(7) & \neg A \vee B \\
(8) & \bot \\
(9) & \neg \neg (\neg A \vee B) \\
(10) & \neg A \vee B
\end{align*}
\]

\[
\begin{align*}
& \Rightarrow E \\
& \Rightarrow I[2] \\
& \Rightarrow E \\
& \Rightarrow I[1] \\
& RAA
\end{align*}
\]

The environment consists of formulae occurring at non-removed leaves.
L’exemple du cours

<table>
<thead>
<tr>
<th>contexte</th>
<th>numero</th>
<th>preuve</th>
<th>justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>Assume $(p \Rightarrow \neg j) \land (\neg p \Rightarrow j) \land (j \Rightarrow m)$</td>
<td>$\wedge E \ 1$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$\neg p \Rightarrow j$</td>
<td>$\wedge E \ 1$</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>$j \Rightarrow m$</td>
<td></td>
</tr>
<tr>
<td>1,4</td>
<td>4</td>
<td>Assume $\neg (m \lor p)$</td>
<td></td>
</tr>
<tr>
<td>1,4,5</td>
<td>5</td>
<td>Assume $p$</td>
<td>$\lor I \ 5$</td>
</tr>
<tr>
<td>1,4,5</td>
<td>6</td>
<td>$m \lor p$</td>
<td>$\Rightarrow E \ 4,6$</td>
</tr>
<tr>
<td>1,4</td>
<td>7</td>
<td>$\bot$</td>
<td>$\Rightarrow I \ 5,7$</td>
</tr>
<tr>
<td>1,4</td>
<td>8</td>
<td>Therefore $\neg p$</td>
<td></td>
</tr>
<tr>
<td>1,4</td>
<td>9</td>
<td>$j$</td>
<td>$\Rightarrow E \ 2, 8$</td>
</tr>
<tr>
<td>1,4</td>
<td>10</td>
<td>$m$</td>
<td>$\Rightarrow E \ 3, 9$</td>
</tr>
<tr>
<td>1,4</td>
<td>11</td>
<td>$m \lor p$</td>
<td>$\lor I \ 10$</td>
</tr>
<tr>
<td>1,4</td>
<td>12</td>
<td>$\bot$</td>
<td>$\Rightarrow E \ 4, 11$</td>
</tr>
<tr>
<td>1</td>
<td>13</td>
<td>Therefore $\neg p (m \lor p)$</td>
<td>$\Rightarrow I \ 4, 13$</td>
</tr>
<tr>
<td>1</td>
<td>14</td>
<td>$m \lor p$</td>
<td>$RAA \ 13$</td>
</tr>
</tbody>
</table>
Plan

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Conclusion
Theorem

Theorem 3.3.1
If a formula $A$ is deduced from an environment $\Gamma$ ($\Gamma \vdash A$) then $A$ is a consequence of $\Gamma$ ($\Gamma \models A$).

Every proof written in an environment $\Gamma$ is correct!
Theorem

Theorem 3.3.1

If a formula $A$ is deduced from an environment $\Gamma$ ($\Gamma \vdash A$) then $A$ is a consequence of $\Gamma$ ($\Gamma \models A$).

Every proof written in an environment $\Gamma$ is correct!

Proof by induction on the number of lines in the proof $P$:

- Let $H_i$ be the context and $C_i$ the conclusion of the $i^{th}$ line in $P$.
- We show that for every $k$ we have $\Gamma, H_k \models C_k$. 
Theorem 3.3.1

If a formula $A$ is deduced from an environment $\Gamma$ ($\Gamma \vdash A$) then $A$ is a consequence of $\Gamma$ ($\Gamma \models A$).

Every proof written in an environment $\Gamma$ is correct!
Proof by induction on the number of lines in the proof $P$:

- Let $H_i$ be the context and $C_i$ the conclusion of the $i^{th}$ line in $P$.
- We show that for every $k$ we have $\Gamma, H_k \models C_k$.

Hence, for the last line $n$ of the proof: $\Gamma \models A$
($H_n$ is empty and $C_n = A$)
Base case

Assume that $A$ is derived from $\Gamma$ by an empty proof.

That is, $A$ is a member of $\Gamma$.

Hence $\Gamma \models A$. 
Induction hypothesis

Assume that for every line $i < k$ of the proof we have $\Gamma, H_i \vdash C_i$.

Let us show that $\Gamma, H_k \vdash C_k$. 
Induction hypothesis

Assume that for every line $i < k$ of the proof we have $\Gamma, H_i \models C_i$.

Let us show that $\Gamma, H_k \models C_k$.

Three possible cases:

- Line $k$ is “Assume $C_k$”.
- Line $k$ is “Therefore $C_k$”.
- Line $k$ is “$C_k$”.
Line \( k \) is "Assume \( C_k \)"

The formula \( C_k \) is the last formula of \( H_k \).

Then \( H_k \models C_k \).

Then \( \Gamma, H_k \models C_k \).
The line $k$ is “Therefore $C_k$”

$C_k$ is the formula $B \Rightarrow D$ where:

- $B$ is the last formula of $H_{k-1}$ and $H_{k-1} = H_k, B$
- $D$ is either a formula in $\Gamma$ or is usable on the previous line.
The line $k$ is “Therefore $C_k$”

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1. If $D$ is a formula of $\Gamma$.

2. If $D$ is usable on the previous line.
The line \( k \) is “Therefore \( C_k \)”

\( C_k \) is the formula \( B \Rightarrow D \) where:

- \( B \) is the last formula of \( H_{k-1} \) and \( H_{k-1} = H_k, B \)
- \( D \) is either a formula in \( \Gamma \) or is usable on the previous line.

1. If \( D \) is a formula of \( \Gamma \).
   \[
   \Gamma, H_k \models D.
   \]
   Since \( D \models B \Rightarrow D \), we conclude that \( \Gamma, H_k \models C_k \).

2. If \( D \) is usable on the previous line.
The line \( k \) is “Therefore \( C_k \)”

\( C_k \) is the formula \( B \Rightarrow D \) where:

- \( B \) is the last formula of \( H_{k-1} \) and \( H_{k-1} = H_k, B \)
- \( D \) is either a formula in \( \Gamma \) or is usable on the previous line.

(1) If \( D \) is a formula of \( \Gamma \).

\[ \Gamma, H_k \models D. \]

Since \( D \models B \Rightarrow D \), we conclude that \( \Gamma, H_k \models C_k. \)

(2) If \( D \) is usable on the previous line.

Hence there exists a line \( i < k \) such that \( D = C_i \) and \( H_i \) is a prefix of \( H_{k-1} \).

By induction hypothesis, \( \Gamma, H_i \models D. \)
The line $k$ is “Therefore $C_k$”

$C_k$ is the formula $B \Rightarrow D$ where:

- $B$ is the last formula of $H_{k-1}$ and $H_{k-1} = H_k, B$
- $D$ is either a formula in $\Gamma$ or is usable on the previous line.

(1) If $D$ is a formula of $\Gamma$.

$\Gamma, H_k \models D$.

Since $D \models B \Rightarrow D$, we conclude that $\Gamma, H_k \models C_k$.

(2) If $D$ is usable on the previous line.

Hence there exists a line $i < k$ such that $D = C_i$ and $H_i$ is a prefix of $H_{k-1}$.

By induction hypothesis, $\Gamma, H_i \models D$.

Since $H_i$ is a prefix of $H_{k-1}$, we have $\Gamma, H_{k-1} \models D$ which can also be written $\Gamma, H_k, B \models D$.

Therefore $\Gamma, H_k \models B \Rightarrow D$. 
Line $k$ is “$C_k$”

$C_k$ is then the conclusion of a rule, whose premises either:

- are usable on the previous line
- or belong to $\Gamma$. 
Line $k$ is “$C_k$”

$C_k$ is then the conclusion of a rule, whose premises either:

- are usable on the previous line
- or belong to $\Gamma$.

We only consider the rule $\land$I, the other cases being similar. $C_k = (D \land E)$ and the premises of the rule are $D$ and $E$. 
Line \( k \) is “\( C_k \)”

\( C_k \) is then the conclusion of a rule, whose premises either:

▶ are usable on the previous line

▶ or belong to \( \Gamma \).

We only consider the rule \( \land I \), the other cases being similar. \( C_k = (D \land E) \) and the premises of the rule are \( D \) and \( E \).

By induction hypothesis, we have:
\[
\Gamma, H_{k-1} \models D \quad \text{and} \quad \Gamma, H_{k-1} \models E.
\]
Since the line \( k \) does not change the hypotheses, we have \( H_{k-1} = H_k \).
Line \( k \) is “\( C_k \)”

\( C_k \) is then the conclusion of a rule, whose premises either:

- are usable on the previous line
- or belong to \( \Gamma \).

We only consider the rule \( \land I \), the other cases being similar. \( C_k = (D \land E) \) and the premises of the rule are \( D \) and \( E \).

By induction hypothesis, we have:
\( \Gamma, H_{k-1} \models D \) and \( \Gamma, H_{k-1} \models E \).

Since the line \( k \) does not change the hypotheses, we have \( H_{k-1} = H_k \).

Finally \( D, E \models D \land E \). Transitively, \( \Gamma, H_k \models C_k \).
Line $k$ is “$C_k$”

$C_k$ is then the conclusion of a rule, whose premises either:

- are usable on the previous line
- or belong to $\Gamma$.

We only consider the rule $\land I$, the other cases being similar. $C_k = (D \land E)$ and the premises of the rule are $D$ and $E$.

By induction hypothesis, we have:

$\Gamma, H_{k-1} \vdash D$ and $\Gamma, H_{k-1} \vdash E$.

Since the line $k$ does not change the hypotheses, we have $H_{k-1} = H_k$.

Finally $D, E \vdash D \land E$. Transitivity, $\Gamma, H_k \vdash C_k$.

For the other rules, it is the same proof, you just have to prove that the conclusion is a consequence of the premises.
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Conclusion
We prove the completeness of the rules only for formulas containing the following logic symbols: \( \bot, \land, \lor, \Rightarrow \).

This is enough because additional symbols \( \top, \neg \) and \( \leftrightarrow \) can be regarded as abbreviations.
Theorem

We prove the completeness of the rules only for formulas containing the following logic symbols: $\bot, \wedge, \vee, \Rightarrow$.

This is enough because additional symbols $\top, \neg$ and $\iff$ can be regarded as abbreviations.

Theorem 3.4.1

Let $\Gamma$ be a finite set of formulae and $A$ a formula. If $\Gamma \models A$ then $\Gamma \vdash A$. 
Definitions

A literal is either a variable $x$ or an implication $x \Rightarrow \bot$.

$x$ and $x \Rightarrow \bot$ (abbreviated as $\neg x$) are complementary literals.
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We define a measure $m$ of formulae and of lists of formulae as:

- $m(\bot) = 0$
- $m(x) = 1$
- $m(A \Rightarrow B) = 1 + m(A) + m(B)$
- $m(A \land B) = 1 + m(A) + m(B)$
- $m(A \lor B) = 2 + m(A) + m(B)$
- $m(\Gamma) = \sum_{A \in \Gamma} m(A)$
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- $m(A \lor B) = 2 + m(A) + m(B)$
- $m(\Gamma) = \sum_{A \in \Gamma} m(A)$

For example, let $A = (a \lor \neg a)$.

$m(\neg a) = 2$, $m(A) = 5$ and $m(A, (b \land b), A) = 13$. 
Induction

We define $P(n)$ to be the following property:
If $m(\Gamma, A) = n$, then if $\Gamma \models A$ then $\Gamma \vdash A$. 

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If $m(\Gamma, A) = n$, then if $\Gamma \models A$ then $\Gamma \vdash A$.

To show that $P(n)$ holds for every integer $n$, we use “strong” induction:
Induction

We define \( P(n) \) to be the following property:
If \( m(\Gamma, A) = n \), then if \( \Gamma \models A \) then \( \Gamma \vdash A \).

To show that \( P(n) \) holds for every integer \( n \), we use “strong” induction:

Assume that for every \( i < k \), property \( P(i) \) holds.
Assume that \( m(\Gamma, A) = k \) and \( \Gamma \models A \).
Let us show that \( \Gamma \vdash A \).
Decomposition

Idea: we decompose $\Gamma, A$ in order to apply the induction hypothesis.
Decomposition

**Idea:** we decompose $\Gamma, A$ in order to apply the induction hypothesis.

$A$ is undecomposable if $A$ is $\bot$ or a variable and $\Gamma$ is undecomposable if $\Gamma$ is a list of literals or contain the formula $\bot$. 
Decomposition

**Idea**: we decompose $\Gamma, A$ in order to apply the induction hypothesis.

$A$ is undecomposable if $A$ is $\bot$ or a variable and $\Gamma$ is undecomposable if $\Gamma$ is a list of literals or contain the formula $\bot$.

We study three cases:

Case 1: *Neither $A$, nor $\Gamma$ is decomposable.*
Decomposition

Idea: we decompose \( \Gamma, A \) in order to apply the induction hypothesis.

A is undecomposable if \( A \) is \( \bot \) or a variable and \( \Gamma \) is undecomposable if \( \Gamma \) is a list of literals or contain the formula \( \bot \).

We study three cases:

Case 1: Neither \( A \), nor \( \Gamma \) is decomposable.

Case 2: \( A \) is decomposable.

We decompose \( A \) in two sub-formulae \( B \) and \( C \).

We obtain \( m(\Gamma, B) < m(\Gamma, A) \) and \( m(\Gamma, C) < m(\Gamma, A) \).
Decomposition

Idea: we decompose \( \Gamma, A \) in order to apply the induction hypothesis.

\( A \) is undecomposable if \( A \) is \( \bot \) or a variable and \( \Gamma \) is undecomposable if \( \Gamma \) is a list of literals or contain the formula \( \bot \).

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Case 2: \( A \) is decomposable.
   We decompose \( A \) in two sub-formulae \( B \) and \( C \).
   We obtain \( m(\Gamma, B) < m(\Gamma, A) \) and \( m(\Gamma, C) < m(\Gamma, A) \).

Case 3: \( \Gamma \) is decomposable.
   We choose a decomposable formula (other than \( x \Rightarrow \bot \)) in \( \Gamma \).
Case 1: neither A, nor Γ are decomposable

Then:

- Γ is a list of literals or contains the formula ⊥.
- A is ⊥ or a variable.
Case 1: neither $A$, nor $\Gamma$ are decomposable

Then:

- $\Gamma$ is a list of literals or contains the formula $\bot$.
- $A$ is $\bot$ or a variable.

(a) If $\bot \in \Gamma$ then $A$ can be derived from $\bot$ by the rule $Efq$. 
Case 1: neither $A$, nor $\Gamma$ are decomposable

Then:

- $\Gamma$ is a list of literals or contains the formula $\bot$.
- $A$ is $\bot$ or a variable.

(a) If $\bot \in \Gamma$ then $A$ can be derived from $\bot$ by the rule $Efq$.
(b) If $\Gamma$ is a list of literals then we have two cases:
Case 1: neither $A$, nor $\Gamma$ are decomposable

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(a) If $\bot \in \Gamma$ then $A$ can be derived from $\bot$ by the rule $Efq$.
(b) If $\Gamma$ is a list of literals then we have two cases:

- $A = \bot$.
  Since $s(\Gamma) \models A$, there are two complementary literals in $\Gamma$.
  Therefore $A$ can be derived from $\Gamma$ by the rule $\Rightarrow E$. 
Case 1: neither $A$, nor $\Gamma$ are decomposable

Then:

- $\Gamma$ is a list of literals or contains the formula $\bot$.
- $A$ is $\bot$ or a variable.

(a) If $\bot \in \Gamma$ then $A$ can be derived from $\bot$ by the rule $Efq$.
(b) If $\Gamma$ is a list of literals then we have two cases:

- $A = \bot$.
  Since $s(\Gamma) \vdash A$, there are two complementary literals in $\Gamma$.
  Therefore $A$ can be derived from $\Gamma$ by the rule $\Rightarrow E$.
- $A$ is a variable.
  Since $\Gamma \models A$:
Case 1 : neither $A$, nor $\Gamma$ are decomposable

Then:
- $\Gamma$ is a list of literals or contains the formula $\bot$.
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  - $A = \bot$.
    Since $s(\Gamma) \models A$, there are two complementary literals in $\Gamma$.
    Therefore $A$ can be derived from $\Gamma$ by the rule $\Rightarrow E$.
  - $A$ is a variable.
    Since $\Gamma \models A$:
    - either $\Gamma$ contains two complementary literals, and similarly $\Gamma \vdash A$
Case 1 : neither $A$, nor $\Gamma$ are decomposable

Then:
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(b) If $\Gamma$ is a list of literals then we have two cases:
  - $A = \bot$.
    Since $s(\Gamma) \models A$, there are two complementary literals in $\Gamma$.
    Therefore $A$ can be derived from $\Gamma$ by the rule $\Rightarrow E$.
  - $A$ is a variable.
    Since $\Gamma \models A$:
      - either $\Gamma$ contains two complementary literals, and similarly $\Gamma \vdash A$
      - or $A \in \Gamma$ and in this case $\Gamma \vdash A$ (by empty proof).
Case 2: \( A \) is decomposable into \( B \) and \( C \)

\( A \) is decomposed into \( B \land C \), \( B \lor C \), or \( B \Rightarrow C \).

We only study the case \( A = B \land C \), the other cases are similar.
Case 2: $A$ is decomposable into $B$ and $C$

$A$ is decomposed into $B \land C$, $B \lor C$, or $B \Rightarrow C$.

We only study the case $A = B \land C$, the other cases are similar.

Since $\Gamma \models A$ and $A = B \land C$, we have $\Gamma \models B$ and $\Gamma \models C$. 
Case 2: $A$ is decomposable into $B$ and $C$

$A$ is decomposed into $B \land C$, $B \lor C$, or $B \Rightarrow C$.

We only study the case $A = B \land C$, the other cases are similar.

Since $\Gamma \models A$ and $A = B \land C$, we have $\Gamma \models B$ and $\Gamma \models C$.

The measures of $B$ and $C$ are strictly less than the measure of $A$, hence $m(\Gamma, B) < k$ and $m(\Gamma, C) < k$. 
Case 2: $A$ is decomposable into $B$ and $C$

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By induction hypothesis, there exist two proofs $P$ and $Q$ such that $\Gamma \vdash P : B$ and $\Gamma \vdash Q : C$. 
Case 2: $A$ is decomposable into $B$ and $C$

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Since $\Gamma \models A$ and $A = B \land C$, we have $\Gamma \models B$ and $\Gamma \models C$.

The measures of $B$ and $C$ are strictly less than the measure of $A$, hence $m(\Gamma, B) < k$ and $m(\Gamma, C) < k$.

By induction hypothesis, there exist two proofs $P$ and $Q$ such that $\Gamma \vdash P : B$ and $\Gamma \vdash Q : C$.

Hence the proof “$P, Q, A$” is a proof of $A$ in the environment $\Gamma$. 
Case 3: \( \Gamma \) is decomposable

There is a decomposable formula in \( \Gamma \) which is either:

- \( B \land C \)
- \( B \lor C \)
- \( B \Rightarrow C \) où \( C \neq \bot \)
- \( (B \land C) \Rightarrow \bot \)
- \( (B \lor C) \Rightarrow \bot \)
- \( (B \Rightarrow C) \Rightarrow \bot \)

We only study the first case.
Γ is a permutation of the list \((B \land C), \Delta\)
Γ is a permutation of the list \((B \land C), \Delta\)

Γ and \((B \land C), \Delta\) have the same measure.
Γ is a permutation of the list \((B \land C), \Delta\)

Γ and \((B \land C), \Delta\) have the same measure.

Since \(\Gamma \models A\), we have \(B, C, \Delta \models A\).
Γ is a permutation of the list \((B \land C), \Delta\)

Γ and \((B \land C), \Delta\) have the same measure.

Since \(\Gamma \vdash A\), we have \(B, C, \Delta \vdash A\).

The sum of the measures of \(B\) and \(C\) is strictly less than the measure of \(B \land C\).
Γ is a permutation of the list $(B \land C), \Delta$

Γ and $(B \land C), \Delta$ have the same measure.

Since $\Gamma \vdash A$, we have $B, C, \Delta \vdash A$.

The sum of the measures of $B$ and $C$ is strictly less than the measure of $B \land C$.

Hence $m(B, C, \Delta, A) < m((B \land C), \Delta, A) = m(\Gamma, A) = k$, by induction hypothesis, there exist a proof $P$ such that $B, C, \Delta \vdash P : A$. 
Γ is a permutation of the list \((B \land C), \Delta\)

Γ and \((B \land C), \Delta\) have the same measure.

Since \(\Gamma \models A\), we have \(B, C, \Delta \models A\).

The sum of the measures of \(B\) and \(C\) is strictly less than the measure of \(B \land C\).

Hence \(m(B, C, \Delta, A) < m((B \land C), \Delta, A) = m(\Gamma, A) = k\), by induction hypothesis, there exist a proof \(P\) such that \(B, C, \Delta \vdash P : A\).

Since \(B\) can be derived from \((B \land C)\) by the rule \(\land E_1\) and \(C\) can be derived from \((B \land C)\) by the rule \(\land E_2\) : “\(B, C, P\)” is a proof of \(A\) in the environment \(\Gamma\).
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Conclusion
Remark 3.4.2

The proof of completeness is **constructive**, that is it provides an **algorithm** to build a proof of a formula in an environment.

However, this algorithm can lead to long proofs.
Remark 3.4.2

The proof of completeness is constructive, that is it provides an algorithm to build a proof of a formula in an environment.

However, this algorithm can lead to long proofs.

The tool
http://teachinglogic.univ-grenoble-alpes.fr/DN/
builds proofs more more efficiently.
Remark 3.4.2

The proof of completeness is constructive, that is it provides an algorithm to build a proof of a formula in an environment.

However, this algorithm can lead to long proofs.

The tool
http://teachinglogic.univ-grenoble-alpes.fr/DN/
builds proofs more more efficiently.
It uses “optimised” tactics presented in section 3.2.

For example, to prove $B \lor C$:

- First try to prove $B$
- If failure, then try to prove $C$
- Otherwise, use tactic 10 (prove $C$ under the hypothesis $\neg B$)
Proof tactics

We wish to prove $A$ in the environment $\Gamma$

The 13 following tactics must be used in the following order!

- Tactics 1 to 3: the proof is over
- Tactics 4 to 6: proof guided by the conclusion (Intro rules)
- Tactics 7 to 9: proof guided by the environment (Elim rules)
- Tactics 10 to 13: reasoning by absurd
Tactic 1

If $A \in \Gamma$ then the empty proof is obtained.
Tactic 2

If $A$ is the consequence of a rule whose premises are in $\Gamma$, then the obtained proof is "$A$".
Tactic 3

If $\Gamma$ contains a contradiction, that is a formula $B$ and a formula $\neg B$, then the obtained proof is “$\bot, A$”.
Tactic 4

If $A$ is $B \land C$ then:

<table>
<thead>
<tr>
<th>contexte</th>
<th>preuve</th>
<th>justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma$</td>
<td>$B$</td>
<td>$\cdots \text{P}\cdots$</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>$C$</td>
<td>$\cdots \text{Q}\cdots$</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>$B \land C$</td>
<td>$\land I$</td>
</tr>
</tbody>
</table>

The proofs can fail (if $\Gamma \not|= A$).

Here, if the proof of $B$ or $C$ fails, the proof of $A$ fails too.

In the remainder of the lecture, we do not highlight the failure cases anymore, unless another proof has to be tried.
Tactic 4

If $A$ is $B \land C$ then:

<table>
<thead>
<tr>
<th>contexte</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma$</td>
<td>$B$</td>
<td>$\ldots P \ldots$</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>$C$</td>
<td>$\ldots Q \ldots$</td>
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</tr>
</tbody>
</table>

The proofs can fail (if $\Gamma \not\models A$).
Here, if the proof of $B$ or $C$ fails, the proof of $A$ fails too.
In the remainder of the lecture, we do not highlight the failure cases anymore, unless another proof has to be tried.
### Tactic 5

If $A$ is $B \Rightarrow C$, then prove $C$ under hypothesis $B$:

<table>
<thead>
<tr>
<th>contexte</th>
<th>preuve</th>
<th>justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma, B$</td>
<td>Assume $B$</td>
<td>[\cdots P \cdots] $\Rightarrow I$</td>
</tr>
<tr>
<td>$\Gamma, B$</td>
<td>$C$</td>
<td></td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>Therefore $B \Rightarrow C$</td>
<td></td>
</tr>
</tbody>
</table>
Tactic 6

If $A$ is $B \lor C$, then prove $B$:

<table>
<thead>
<tr>
<th>contexte</th>
<th>preuve</th>
<th>justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma$</td>
<td>$B$</td>
<td>$\cdots P \cdots$</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>$B \lor C$</td>
<td>$\lor I 1$</td>
</tr>
</tbody>
</table>

If the proof of $B$ fails then prove $C$:

<table>
<thead>
<tr>
<th>contexte</th>
<th>preuve</th>
<th>justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma$</td>
<td>$C$</td>
<td>$\cdots P \cdots$</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>$B \lor C$</td>
<td>$\lor I 2$</td>
</tr>
</tbody>
</table>

If the proof of $C$ fails, try the following tactics.
Tactic 7

If $B \land C$ is in the environment, then prove $A$ starting from formulae $B$, $C$, replacing $B \land C$ in the environment:

<table>
<thead>
<tr>
<th>contexte</th>
<th>preuve</th>
<th>justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma, B \land C$</td>
<td>$B$</td>
<td>$\land E1$</td>
</tr>
<tr>
<td>$\Gamma, B \land C$</td>
<td>$C$</td>
<td>$\land E2$</td>
</tr>
<tr>
<td>$\Gamma, B \land C$</td>
<td>$A$</td>
<td>...$P$...</td>
</tr>
</tbody>
</table>
Tactic 8

If $B \lor C$ is in the environment, then:

- prove $A$ in the environment where $B$ replaces $B \lor C$.
- prove $A$ in the environment where $C$ replaces $B \lor C$.

<table>
<thead>
<tr>
<th>contexte</th>
<th>preuve</th>
<th>justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma, B \lor C, B$</td>
<td>Assume $B$</td>
<td>$\ldots P \ldots$</td>
</tr>
<tr>
<td>$\Gamma, B \lor C, B$</td>
<td>$A$</td>
<td>$\Rightarrow I$</td>
</tr>
<tr>
<td>$\Gamma, B \lor C$</td>
<td>Therefore $B \Rightarrow A$</td>
<td></td>
</tr>
<tr>
<td>$\Gamma, B \lor C, C$</td>
<td>Assume $C$</td>
<td>$\ldots Q \ldots$</td>
</tr>
<tr>
<td>$\Gamma, B \lor C, C$</td>
<td>$A$</td>
<td>$\Rightarrow I$</td>
</tr>
<tr>
<td>$\Gamma, B \lor C$</td>
<td>Therefore $C \Rightarrow A$</td>
<td>$\lor E$</td>
</tr>
<tr>
<td>$\Gamma, B \lor C$</td>
<td>$A$</td>
<td></td>
</tr>
</tbody>
</table>
Tactic 9

If $\neg(B \lor C)$ is in the environment, then

- we derive $\neg B$ by the proof $P_4$ and
- $\neg C$ by the proof $P_5$ (proofs requested in exercise 57).

Let $P$ the proof of $A$ in the environment where $\neg B$, $\neg C$ replace the formula $\neg(B \lor C)$.

The proof of $A$ is “$P_4$, $P_5$, $P$”.
Tactic 10

If $A$ is $B \lor C$, then prove $C$ under hypothesis $\neg B$: let $P$ the obtained proof.

“Assume $\neg B$, $P$, Therefore $\neg B \Rightarrow C$” is a proof of the formula $\neg B \Rightarrow C$ which is equivalent to $A$.

To obtain the proof of $A$, simply add the proof $P1$, requested in exercise 57 of $A$ in the environment $\neg B \Rightarrow C$.

The proof obtained from $A$ is therefore “Assume $\neg B$, $P$, Therefore $\neg B \Rightarrow C$, $P1$”.
Tactic 11

If \( \neg (B \land C) \) is in the environment, then we deduce from it \( \neg B \lor \neg C \) by the proof \( P3 \) requested in exercise 57 then we reason case by case as follows:

1. prove \( A \) in the environment where \( \neg B \) replaces \( \neg (B \land C) \): Let \( P \) the obtained proof,
2. prove \( A \) in the environment where \( \neg C \) replaces \( \neg (B \land C) \): Let \( Q \) the obtained proof.

The proof of \( A \) is “\( P3, \text{Assume } \neg B, P, \text{Therefore } \neg B \Rightarrow A, \text{Assume } \neg C, Q, \text{Therefore } \neg C \Rightarrow A, A \)”.
Tactique 12

If $\neg(B \Rightarrow C)$ is in the environment, then

- we derive $B$ by the proof $P6$,
- $\neg C$ by the proof $P7$ (proofs requested in exercise 57).
- Let $P$ the proof of $A$ in the environment where $B$, $\neg C$ replace the formula $\neg(B \Rightarrow C)$.

The proof of $A$ is “$P6, P7, P$”.
Tactic 13

If \( B \Rightarrow C \) is in the environment and if \( C \neq \bot \), i.e. if \( B \Rightarrow C \) is not \( \neg B \), then,

we derive \( \neg B \vee C \) in the environment \( B \Rightarrow C \) by proof \( P2 \) from exercise 57, then we reason by cases:

> prove \( A \) in the environment where \( \neg B \) replaces \( B \Rightarrow C \): Let \( P \) the obtained proof,

> prove \( A \) in the environment where \( C \) replaces \( B \Rightarrow C \): Let \( Q \) the obtained proof.

The proof of \( A \) is \( "P2, Assume \neg B, P, Therefore \neg B \Rightarrow A, Assume C, Q, Therefore C \Rightarrow A, A". \)
Example

Proof of Peirce’s formula:

\[ ((p \Rightarrow q) \Rightarrow p) \Rightarrow p \]
Proof plan

Tactic 5 is compulsory!

Proof $Q$:
Assume $(p \Rightarrow q) \Rightarrow p$

$Q_1$ proof or $p$ in the environment $(p \Rightarrow q) \Rightarrow p$

Therefore $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$

Proof $Q_1$ necessarily uses tactic 13 (the environment is $B \Rightarrow C = (p \Rightarrow q) \Rightarrow p$).

Hence we have to prove $p$ both:

- in the environment $\neg B = \neg (p \Rightarrow q)$
- in the environment $C = p$. 
Plan of the proof of $Q_1$

Proof $Q_1$:

$Q_{11}$ is the proof of $\neg B \vee C$ in the environment $B \Rightarrow C$, see exercise 57

Assume $\neg (p \Rightarrow q)$

$Q_{12}$ proof of $p$ in the environment $\neg (p \Rightarrow q)$

Therefore $\neg (p \Rightarrow q) \Rightarrow p$

Assume $p$

$Q_{13}$ proof of $p$ in the environment $p$

Therefore $p \Rightarrow p$

$p$
Proof of $Q_1$

$Q_{13}$ is the empty proof, since $A = C = p$.

$Q_{12}$ is the proof of $C = p$ in the environment $\neg(p \Rightarrow q)$. Since $\neg A$ is an abbreviation of $A \Rightarrow \bot$, this proof is the proof $P_6$ requested in exercise 57, where $B = p$ and $C = q$.

By gluing pieces $Q_1$, $Q_{11}$, $Q_{12}$, $Q_{13}$, we obtain the proof $Q$.

The proof $Q_{12}$ can also be done without using the tactics.
Plan

Correctness

Completeness

Tactics

Conclusion
Today

- Propositional Natural Deduction is correct and complete.
- Tactics for building a proof
Automated proofs

To automatically obtain the proofs in the system, we recommend the following software (implementing the 13 previous tactics):

http://teachinglogic.univ-grenoble-alpes.fr/DN/

People who prefer tree-like proofs can use the following software:

http://www-sop.inria.fr/marelle/Laurent.Thery/peanoware/Nd.html
Overview of the Semester

TODAY

▸ Propositional logic
▸ Propositional resolution
▸ Natural deduction for propositional logic *
▸ First order logic

MIDTERM EXAM

▸ Logical basis for automated proving
  (“first-order resolution”)
▸ First-order natural deduction

EXAM